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Testing for marginal asymmetry of weakly dependent processes

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Testing for marginal asymmetry of weakly dependent processes[1]

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Abstract

This article addresses the issue of testing for asymmetry of the marginal law of weakly dependent processes. A modified quantile-based symmetry test is considered. The test has an intuitive interpretation, it is easy and fast to calculate, follows a standard limiting distribution, and much importantly, it is robust against weak dependence of observations and outliers. The finite sample performance of the robust test is examined via Monte Carlo experiments. An empirical application using economic indicators is provided as well.

JEL classification: C12, C14, C15, C22
Key words: marginal symmetry, sample quantiles, Monte Carlo experiments

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1. Introduction

Symmetry of the marginal law of random variables is a fundamental condition for many statistical methods[3] and theoretical concepts in economics[4] and finance[5]. However, testing for asymmetry in economic time series, which can be considered as weakly dependent stochastic processes, is by no means easy in practice. Two problems immediately arise.

First, economic time series do exhibit some form of weak dependence, which invalidates critical values of standard symmetry tests originally derived for independently and identically distributed (IID) random variables. Only quite recently, some symmetry tests have been developed for weakly dependent (WD) stochastic processes as well. Bai and Ng (2005) proposed a symmetry test based on sample skewness, whereas Psaradakis (2003) developed a Kolmogorov-Smirnov test based on a sieve bootstrap. However, it is important to point out that both tests suffer from some shortcomings and their application to economic time series might be problematic. The main drawbacks of the tests are related to either the problematic estimation of some key quantities or strong assumptions about the underlying stochastic process under consideration. For example, it is well known that the estimation of higher-order moments (e.g. the variance-covariance matrix) is more involved for weakly dependent processes. As shown in Andrews (1991), standard variance-covariance estimators perform very poorly for (persistent) WD stochastic processes. As a result, the Bai and Ng (2005) symmetry test suffers from a significant power loss, which, in turn, can lead to misleading inference. The problem with

[3]Since the influential work of Sims (1980), VAR models have become a popular tool for modelling and forecasting of economic variables. Nevertheless, provided that stochastic processes do exhibit marginal asymmetry, routinely applied linear Gaussian VAR models cannot adequately describe the main stochastic features of economic variables. In addition, provided that the violation of the symmetry condition is due to asymmetry in model innovations, then the usually used quasi maximum likelihood estimator (QMLE) leads to inefficiency and possible inconsistency of the estimated parameters, see Newey and Steigerwald (1997) for theoretical arguments, and Tiku et al. (2001) and Pope (1990) for Monte Carlo evidence. Such model misspecification gives, in turn, very likely rise to misleading inference (e.g. biased point forecasts, invalid impulse response functions, or misleading forecast error variance decompositions, etc).

[4]Since the Great depression in 1930's, causes and consequences of business cycle fluctuations have attracted much interest in theoretical and empirical macroeconomics. As for economic theory, the presence of asymmetry in economic indicators has implications for developing theoretical models in economics. For example, Acemoglu and Scott (1997) build a model where business cycle fluctuations are based on intertemporal increasing returns in the economy. They show that this model specification is helpful in explaining business fluctuations even in the case of independent and identically distributed shocks because individuals respond differently to shocks depending on their past experience. As for empirical economics, Boldin (1999) shows that the effect of monetary policy measures varies over the business cycle. As a result, ignoring business cycle asymmetry can lead to misleading economic policy conclusions.

[5]A commonly applied option pricing model in finance is a famous Black-Scholes formula developed by Black and Scholes (1973). The pricing model explicitly assumes that equity returns are marginally normally distributed, and, therefore, symmetric. If this underlying assumption fails, for instance, due to the presence of asymmetry in equity returns, then the BS formula very likely misprices options. Therefore, some modifications of the BS formula have been developed in the literature. The modifications usually rely on a Gram-Charlier or Edgeworth expansion of some flexible probability distribution, see Jarrow and Rudd (1982) and Corrado and Su (1996), among others. However, as shown by Vahamaa (2003), more sophisticated pricing formulae can produce even larger pricing errors as compared to the original BS formula, mainly due to the numerical difficulties related to more sophisticated methods.

the Kolmogorov-Smirnov test proposed in Psaradakis (2003) is that a sieve bootstrap is theoretically valid only for a limited class of linear models allowing for an AR(∞) representation. Another problem is that the procedure might not give satisfactory results for non-IID innovations (e.g. martingale difference sequences) in general. Although the bootstrap-based test gives very satisfactory results even in small samples (i.e. $T = 100$), and clearly outperforms the Bai and Ng test, it is computationally intensive.

Second, economic time series are often contaminated by outliers, see Balke and Fomby (1994) for empirical evidence. The problem is that standard symmetry tests are not robust against outliers, which means that a sufficiently large aberrant observation biases the measure of asymmetry (e.g. the coefficient of skewness), and, thus, can lead to misleading inference, see Bowman and Shenton (1975) for a discussion, and Peiró (1999) or Premaratne and Bera (2005) for Monte Carlo evidence.

This paper brings three contributions to the literature of testing for symmetry of weakly dependent processes. First, a modified test of symmetry based on sample quantiles is discussed in this paper. The test has an intuitive interpretation, it is easy to calculate, it has a standard limiting distribution, and it is robust against outliers and weak dependence of observations. Especially the last feature is very useful for applied research. It will be shown later on in the paper that the quantile-based specification of the test makes the computation of some key quantities (e.g. the variance-covariance matrix) insensitive to dependence of observations. This fact significantly reduces possibility of inferential errors caused by the incorrect configuration and implementation of the test. Second, the proposed quantile-based symmetry test requires the existence of only the first four moments of a given random variable. This is a significant improvement as compared to the Bai and Ng test, whose symmetry test requires the existence of the first six moments.[6] As a result, the proposed quantile-based symmetry test reduces the probability of misleading inference due to moment condition failure. Third, Monte Carlo experiments suggest that the finite sample performance of the quantile-based test is very good. It will be shown that the size and power properties of the quantile-based test are comparable to a more advanced and computationally intensive bootstrap-based Kolmogorov-Smirnov test of symmetry It means that in situations where the user is not proficient in a bootstrap, or it is not clear which bootstrap method should be implemented, the quantile-based test may serve as a valuable alternative.

The paper is organized as follows. Two symmetry tests are discussed in Section 2. Monte

[6]Note that such a moment requirement seems to be too high and in sharp contrast with empirical findings about some economic time series (e.g equity returns, exchange rate returns, or interest rates). The interested reader is referred to Runde (1997), Koedijk et al. (1990), and Bali (2003), for empirical evidence about the existence of moments.

Carlo setup and results are discussed in Section 3. Finally, testing symmetry of economic time series is discussed in Section 4.

2. Assumptions and test

2.1 Robust measure of skewness

Let $\{Y_t : t \in \mathbb{Z}\}$ be a strictly stationary sequence of random variables with the marginal distribution function $F(y) = \mathbb{P}(Y - 1 \leq y)$, for any $y \in \mathbb{R}$. The problem of interest is to test the hypothesis that F is symmetric about ζ, the centre of symmetry, that is,

$$\mathbb{F}(\zeta + y) = 1 - \mathbb{F}(\zeta - y), \quad \text{for every} \quad y \in \mathbb{R}, \zeta \in \mathbb{R}. \tag{1}$$

The test for symmetry explored here relies on a measure of skewness based on selected quantiles of F. Specifically, letting $\xi_p = \inf\{y : F(y) \geq p\}$, $p \in (0, 1)$, denote the p-th quantile of F, it is easy to see that $\xi_{1/2} - \xi_p = \xi_{1-p} - \xi_{1/2}$ when (1) holds. Motivated by this observation, we consider the following measure of skewness

$$S = \boldsymbol{\delta}'\boldsymbol{\xi}, \tag{2}$$

where $\boldsymbol{\xi} = (\xi_{p_1}, \ldots, \xi_{p_k}, \xi_{1/2}, \xi_{1-p_k}, \ldots, \xi_{1-p_1})'$ for some fixed integer $k \geq 1$ and constants $0 < p_1 \ldots < 1$, and $\boldsymbol{\delta} \neq 0$ is a $(2k + 1 \times 1)$ is fixed selection vector such that $S = 0$ when F is symmetric.[7] Note that for $k = 1$ and $\boldsymbol{\delta} = (1, -2, 1)'$, S becomes an unscaled version of the measure of skewness considered in Hinkley (1975).

It is easy to show that the proposed measure of symmetry S satisfies basic properties discussed in Groeneveld and Meeden (1984). In particular, the index of skewness $S \equiv S(F)$ of a distribution F defined as in (2) satisfies the following properties: (i) $S(F) = S(aF + b)$ for any fixed $b \in \mathbb{R}$ and $a > 0$; (ii) $S(F) = 0$ is F is symmetric; (iii) $S(-F) = -S(F)$; (iv) $S(F) \leq S(G)$ for any distribution G that is at least as skewed to the right as F (i.e. any G distribution such that $G^{-1}(F(y))$ is convex). The verification of the first three properties is quite straightforward, while the verification of the last property ("skew dominance") is more complex, see Groeneveld and Meeden (1984) for a discussion.

Before we proceed to the testing procedure, let us state some necessary assumptions.

Assumption 1 *The process $\{Y_t : t \in \mathbb{Z}\}$ is assumed to be strictly stationary real-valued α-mixing such that $\alpha(n) = O(n^{-\varphi})$, where $\varphi > 3$, and some integer n such that $n \to \infty$.* □

[7] For example, $\boldsymbol{\delta}_{k+1} = -2$ and $\boldsymbol{\delta}_i = 1/k$ for $i \neq k + 1$, with $\boldsymbol{\delta}_i$ denoting the i-th component of a vector $\boldsymbol{\delta}$.

The strong-mixing coefficients $\alpha(n)$, for some integer n such that $n \to \infty$, of a strictly stationary random sequence $\{Y_t : t \in \mathbb{Z}\}$ are defined as follows

$$\alpha(n) = \sup_{\mathcal{A} \in \mathcal{F}_{-\infty}^{0}, \mathcal{B} \in \mathcal{F}_{n}^{\infty}} |\mathbb{P}(\mathcal{A} \cap \mathcal{B}) - \mathbb{P}(\mathcal{A})\mathbb{P}(\mathcal{B})|, \quad \text{for} \quad n \in \mathbb{N},$$

where, for $-\infty \leq r \leq s \leq \infty$, $\mathcal{F}_r^s$ denotes σ-algebra generated by $\{Y_t : r \leq t \leq s\}$. If $\alpha(n) \to 0$ as $n \to \infty$, then the sequence $\{Y_t : t \in \mathbb{Z}\}$ is said to be α-mixing. The α-mixing assumption is an important condition for the validity of the central limit theorem of quantities calculated from weakly dependent observations, see Lehmann (1999, Ch. 2.8) for details. The α-mixing condition is fairly mild and is satisfied by a wide variety of linear and non-linear random processes. Examples of α-mixing processes include, among many others, Markov chains satisfying mild regularity conditions, non-linear processes admitting a Haris ergodic Markovian representation, ARCH type and stochastic volatility processes, q-dependent processes, and linear processes driven by innovations having a continuous distribution which satisfies suitable smoothness conditions, see Doukhan (1994).

Given a sample $\{Y_1, \ldots, Y_T\}$, for $T \geq 1$, a natural estimator of ξ_p is the p-th sample quantile $\hat{\xi}_p = \inf\{y : \hat{F}(y) \geq p\}$, $p \in (0,1)$, where $\hat{F}(y) = (1/T)\sum_t^T I(Y_t \leq y)$, $y \in \mathbb{R}$, is the sample distribution function and $I(\cdot)$ denotes the indicator function. Under α-mixing condition, it is straightforward to show that $\boldsymbol{\xi}$ is consistently estimated by the vector of sample quantiles $\hat{\boldsymbol{\xi}} = (\hat{\xi}_{p_1}, \ldots, \hat{\xi}_{p_k}, \hat{\xi}_{1/2}, \hat{\xi}_{1-p_k} \ldots, \hat{\xi}_{1-p_1})'$. More specifically, putting $\mathcal{P}_k = \{p_1, \ldots, p_k, 1-p_1, \ldots, 1-p_k, 1/2\}$, we have the following limiting result.

Theorem 1 *Suppose that $\alpha(n) \to 0$ as $n \to \infty$ and that, for every $p \in \mathcal{P}_k$, ξ_p is the unique p-th quantile of F. Then, $\hat{\boldsymbol{\xi}} - \boldsymbol{\xi} \xrightarrow{a.s.} 0$ as $T \to \infty$.* □

Proof. *See Appendix A for the proof.* ■

By strengthening the mixing condition somewhat and imposing some smoothness on the marginal distribution of $\{Y_t\}$, the limiting distribution of $\hat{\boldsymbol{\xi}}$ can be obtained.

Assumption 2 *The distribution function F and the density function f of a stochastic process $\{Y_t : t \in \mathbb{Z}\}$ satisfy the following properties*

(a) F posses a positive density f satisfying a uniform Lipschitz condition of order 1 on $\mathbb{R}$,

(b) $f(u)$ is continuous at any point $u \in \mathbb{R}$,

(c) $f(u) > 0$ *at any point* $u \in \mathbb{R}$ *and bounded,*

(d) $\int_{\mathbb{R}} |f(u)|\mathrm{d}u < \infty$*,*

(e) $f^{(1)}(u)$ *and* $f^{(2)}(u)$ *are continuous and bounded in some neighborhood of* u*,*

(f) $\mathbb{E}(|Y_t|^4) < \infty$ *for all* $t \in \mathbb{Z}$*.* □

The conditions are common restrictions imposed on a density function ensuring that a density function is continuous, non-negative and bounded and having bounded derivatives. Assumption 2 is common in this type of settings and not overly restrictive.

Theorem 2 *Suppose that* $\sum_{n=1}^{\infty} \alpha(n) < \infty$ *and that, for every* $p \in \mathcal{P}_k$*,* F *is differentiable at* ξ_p *with* $F'(\xi_p) = f(\xi_p)$*. Then* $\sqrt{T}(\hat{\boldsymbol{\xi}} - \boldsymbol{\xi}) \xrightarrow{d} N(\mathbf{0}, \boldsymbol{\Sigma})$ *as* $T \to \infty$*, where* $\boldsymbol{\Sigma} = [\sigma_{i,j}]_{i,j=1}^{2k+1}$ *with*

$$\sigma_{i,j} = \frac{1}{f(\boldsymbol{\xi}_i)f(\boldsymbol{\xi}_j)} \left\{ \gamma_{i,j}(0) + \sum_{h=1}^{\infty} [\gamma_{i,j}(h) + \gamma_{j,i}(h)] \right\}, \tag{3}$$

where $\gamma_{i,j}(h) = cov\left[I(Y_1 \leq \boldsymbol{\xi}_i), I(Y_{1+h} \leq \boldsymbol{\xi}_j)\right]$*, for* $h \geq 0$*.* □

Proof. *See Appendix A for a proof.* ■

The differentiability condition on F in Theorem 2 is standard in the literature on sample quantiles and not overly restrictive. In fact, asymptotic normality does not hold if F is not differentiable at the quantiles of interest, see Theorem 2 in Sharipov and Wendler (2013).[8] Note also that the summability condition is the best currently available condition for the central limit theorem for bounded random variables.[9]

2.2 TEST FOR SYMMETRY

Since the measure of skewness S given in (2) is 0 when F satisfies (1), our statistic for testing the hypothesis of marginal symmetry of $\{Y_t\}$ is defined as

$$QS = T\left(\frac{(\hat{S} - S)^2}{\mathrm{var}(\hat{S})}\right) = T(\boldsymbol{\delta}'\hat{\boldsymbol{\xi}})^2/\boldsymbol{\delta}'\hat{\boldsymbol{\Sigma}}\boldsymbol{\delta}, \tag{4}$$

where $\hat{\boldsymbol{\Sigma}}$ is a suitable estimator of $\boldsymbol{\Sigma}$. The following result is an immediate consequence of Theorem 2, the continuous mapping theorem, and Slutsky's theorem.

[8]Non-Gaussian weak limits are assumed to be expected for extreme sample quantiles ξ_p with $p \to 0$ or $p \to 1$, see Beirlant (2004).

[9]It is interesting to note that, as in the case of IID data, the error of approximation in the central limit theorem for sample quantiles is of order $O(1/\sqrt{T}$ under suitable polynomial α-mixing condition, see Lahiri and Sun (2009).

Corollary 1 *Suppose that the conditions in Theorem 2 hold and let* $\hat{\mathbf{\Sigma}}$ *be a consistent estimator of* $\mathbf{\Sigma}$*. Then,* $QS \xrightarrow{d} \chi^2(1)$ *as* $T \to \infty$ *under (1).* □

Proof. *See Appendix A for a proof.* ■

To make the test operational, a consistent estimator of the asymptotic variance-covariance matrix $\mathbf{\Sigma}$ is required in (4). Note that the variance-covariance matrix $\mathbf{\Sigma}$ is not a diagonal matrix even when we deal with IID observations. Following the literature on estimation of asymptotic covariance matrices in the presence of weak dependence, we consider here an estimator $\mathbf{\Sigma} = [\sigma_{i,j}]_{i,j=1}^{2k+1}$ with

$$\hat{\sigma}_{i,j} = \frac{1}{\hat{f}(\hat{\boldsymbol{\xi}}_i)\hat{f}(\hat{\boldsymbol{\xi}}_j)} \left\{ \hat{\gamma}_{i,j}(0) + \sum_{h=1}^{T-1} w(h/m) \left[\hat{\gamma}_{i,j}(h) + \hat{\gamma}_{j,i}(h) \right] \right\}, \tag{5}$$

with

$$\hat{\gamma}_{i,j}(h) = \frac{1}{T} \sum_{t=1}^{T-h} I(Y_t \leq \hat{\boldsymbol{\xi}}_i) I(Y_{t+h} \leq \hat{\boldsymbol{\xi}}_j) - \frac{1}{T^2} \sum_{t=1}^{T-h} I(Y_t \leq \hat{\boldsymbol{\xi}}_i) \sum_{t=1}^{T-h} I(Y_{t+h} \leq \hat{\boldsymbol{\xi}}_j), \tag{6}$$

where $0 \leq h < T$, $w(\cdot)$ are kernel weights, m is a real-valued bandwidth such that $m \to \infty$ and $m/T \to 0$ as $T \to \infty$, and $\hat{f}$ is a consistent estimator of f. Assuming f is a bounded density for F, it is estimated by means of a standard Parzen-Rosenblatt estimator

$$\hat{f}(y) = \frac{1}{bT} \sum_{t=1}^{T} K\left(\frac{y - Y_t}{b}\right), \quad y \in \mathbb{R}, \tag{7}$$

where $K(\cdot)$ is a kernel function and $b > 0$ is a bandwidth such that $b \to 0$ and $Tb \to \infty$ as $T \to \infty$.

In view of Theorems 1 and 2, consistency of the estimator in (5) follows from well-known resuts on covariance matrix estimation (e.g. Andrews (1991, Theorem 1), Hansen (1992, Theorem 2), and De Jong (2000, Theorem 2), among others), combined with uniform consistency of the kernel estimator $\hat{f}$. Since $I(T_t \leq \boldsymbol{\xi}_i)$ is bounded, a mixing rate $\alpha(n) = O(n^{-\beta})$ for some $\beta > r/2$ and $r \in (2, 4]$, coupled with $m = o(T^{1/2-1/r})$, is sufficient for $\hat{\gamma}_{i,j}(0) + \sum_{h=1}^{T-1} w(h/m) \left[\hat{\gamma}_{i,j}(h) + \hat{\gamma}_{j,i}(h) \right]$ to converge in probability to $\gamma_{i,j}(0) + \sum_{h=1}^{\infty} \left[\gamma_{i,j}(h) + \gamma_{j,i}(h) \right]$, see De Jong (2000). Regular conditions which ensure uniform consistency of the kernel estimator in (7) can be found in Cai and Roussas (1992, Theorem 4.1), Liebscher (1996, Theorem 4.2), and Hansen (2008, Theorem 6), among others. A polynomial mixing rate $\alpha(n) = O(n^{\varphi})$ with $\varphi > 3$ and some smoothness conditions for f are typically sufficient for such results to hold. A set of additional conditions under which $\hat{f}(\hat{\boldsymbol{\xi}}_i)$ converges almost surely to $f(\boldsymbol{\xi}_i)$ is stated in the following assumption.

Assumption 3 *The kernel function $K(\cdot)$ is assumed to possess the following properties:*

(a) $K(u) \leq 0$ *for any* $u \in \mathbb{R}$ *and differentiable,*

(b) $\int_{\mathbb{R}} K(u)\mathrm{d}u = 1$*,*

(c) $\int_{\mathbb{R}} |u|K(u)\mathrm{d}u < \infty$*,*

(d) $\int_{\mathbb{R}} |K'(u)|\mathrm{d}u < \infty$*,*

(e) $\lim_{|u|\to\infty} K(u) = 0$*,*

(f) as $T \to \infty$*, the bandwidth parameter* b *is assumed to be a positive quantity such that: (i)* $b \to 0$*; (ii)* $b^2T/\log\log T \to \infty$*.* □

Under Assumptions 1–3, it holds that $\hat{f}(y) - f(y) \xrightarrow{a.s.} 0$ as $T \to \infty$, uniformly in $y \in \mathbb{R}$, by Theorem 4.1 of Cai and Roussas (1992). Together with the result in Theorem 1, this implies that $\hat{f}(\hat{\xi}_p) - f(\xi_p) \xrightarrow{a.s.} 0$ as $T \to \infty$ for every $p \in \mathcal{P}_k$.

Needless to say, there are many choices available for suitable kernels $w(\cdot)$ and $K(\cdot)$ that may be used in (5) and (7). The differences in the resulting estimators are not generally substantial in finite samples (see, e.g., Andrews (1991), Silverman (1986)), and we will take $w(\cdot)$ and $K(\cdot)$ in the remainder of this paper to be the Bartlett kernel and the Gaussian kernel, respectively, i.e., $w(x) = (1 - |x|)I(|x| \leq 1)$ and $K(x) = \exp(-x^2/2)/\sqrt{2\pi}$. With regard to the bandwidth parameters m and b, the former will be selected by means of automatic data-dependent method of Newey and West (1994). For the latter, we will use the popular normal reference bandwidth $b = 0.79(\xi_{3/4} - \xi_{1/4})T^{-1/5}$ discussed in Silverman (1986, Section 3.4.2).[10]

Finally, we note that, instead of relying on covariance estimators of the type given in (5), a bootstrap estimator of the asymptotic covariance matrix $\boldsymbol{\Sigma}$ could be used. Sun and Lahiri (2006) showed that, under a polynomial α-mixing rate and mild smoothness conditions on F, the asymptotic variance of a sample quantile can be consistently estimated by a blockwise bootstrap method. The blockwise bootstrap may also be used to obtain a consistent estimator of the distribution of a sample quantile, see Sun and Lahiri (2006) and Sharipov and Wendler

[10] Of course, many other methods for selecting the bandwidth b are available in the literature (see, e.g., Jones et al. (1996). We found in our simulations that the finite-sample properties of the test for symmetry based on the QS test are fairly robust with respect to different data-dependent bandwith selection methods, and so we focus here on the computationally simple normal reference bandwidth selector. It is also worth noting that bandwidth selectors designed for IID data often work equally well under dependence (see, e.g., Hall et al. (1995).

(2013), among others. Such techniques could be adapted to the problem of testing symmetry using the statistic in (4), but bootstrap-based versions of our test will not be investigated here.

2.3 Simple QS test

One of the main tasks of this paper is to show that the proposed quantile-based symmetry test is robust against weak dependence of observations observed in economic time series. An intuitive explanation can be found in Figure 1. The figure depicts selected sample quantiles from the empirical distribution function (EDF) and the corresponding points in the realization of a given stochastic process (i.e. the DGP is an AR(1) process with standard Gaussian innovations). The robustness of the test follows from the fact that individual sample quantiles are well separated over time, and, thus, might be considered as quantities from an "almost" uncorrelated stochastic process. This example helps to understand why it might be possible to approximate the variance-covariance matrix for WD processes by the variance-covariance matrix for IID processes. The figure also clearly shows the trade-off between the number of quantiles and their dependence: the more quantiles the higher the dependence.[11] In order to explicitly demonstrate the robustness of the test, a modified symmetry test, denoted as QS*, is considered as well. The test statistic is given by

$$QS^{*} = T(\boldsymbol{\delta}'\hat{\boldsymbol{\xi}})^2/\boldsymbol{\delta}'\hat{\boldsymbol{\Sigma}}^{*}\boldsymbol{\delta}, \tag{8}$$

where $\hat{\boldsymbol{\Sigma}}^{*} = [\sigma^{*}_{i,j}]_{i,j=1}^{2k+1}$ is the estimated variance-covariance matrix for IID observations given by

$$\hat{\sigma}^{*}_{i,j} = \frac{p_i(1-p_j)}{\hat{f}(\hat{\boldsymbol{\xi}}_i)\hat{f}(\hat{\boldsymbol{\xi}}_j)}, \quad \text{for} \quad i \leq j. \tag{9}$$

The estimation of $\hat{f}$ is described in the previous section. Recall that the modified variance-covariance matrix is strictly correct for IID observations and only an approximation for WD observations.

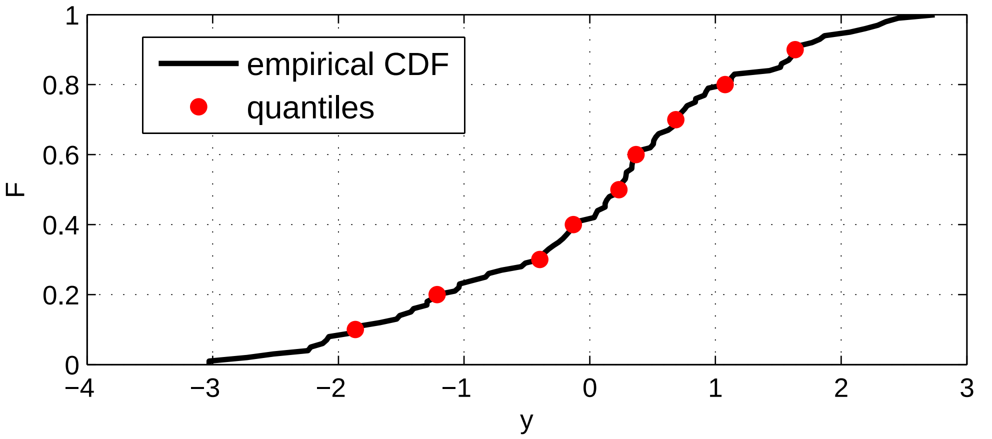
3. Monte Carlo setup and results

3.1 Monte Carlo setup

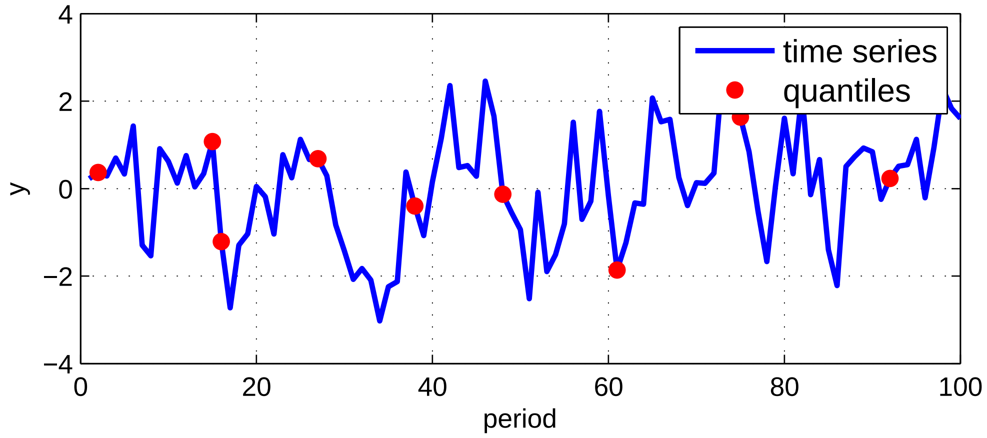
The size and power properties of the proposed quantile symmetry test are assessed using a list of time series models usually used in applied economics/finance: (i) M1–M3 models represent autoregressive and moving average (ARMA) models; (ii) H1–H2 models denote autoregressive conditional heteroscedasticity (GARCH) models with a different functional form; (ii) N1–N2

[11] We leave the issue of optimal number of quantiles for further research.

Figure 1: Empirical quantiles



(a) empirical CDF and estimated quantiles



(b) time series and estimated quantiles

Note: the data generating process is $y_t = 0.5y_{t-1} + a_t$, where $a \sim NID(0, 1)$.

models denote mixture autoregressive (MAR) models with a different functional form. A complete set of models can be found in Table 1.

Since various DGPs are considered in this paper, some comments are in order. The marginal distribution of ARMA models is asymmetric, provided that the distribution of innovations is asymmetric. Asymmetry of the marginal distribution of GARCH processes is caused by asymmetry of model innovations.[12] In contrast, asymmetry in the marginal law of MAR models can be caused by a combination of different model parameters such as regime constants, autoregressive parameters, and regime variances. Note that all the above mentioned time series models satisfy the necessary strong-mixing condition under relatively mild assumptions, see Chanda (1974, p. 403) for ARMA models and Francq and Zakoïan (2006, p. 822) for GARCH,

[12] Note, however, that the closed form solution of the marginal distribution is not known for GARCH processes in general. The closed-form distribution is available only for specific GARCH-type models such as moving average conditional heteroscedastic processes, see Yang and Bewley (1995) for details.

Liebscher (2005, p. 680) for NLAR (and therefore MAR models as well) models. In order to meet the moment condition of the test, restrictions on some parameters of DGPs must be imposed. In particular, we set the model parameters and the distributions of innovations in such a way that $\mathbb{E}(|Y_t|^4) < \infty$.

The power properties of the proposed robust symmetry tests are examined on various distributions of innovations. In particular, apart from a Gaussian distribution, which serves as a benchmark for comparison, we consider model innovations coming from a generalized lambda distributions (GLD), see Randles et al. (1980). This family provides a wide range of distributions that are easily generated, since they are defined in terms of the inverse of the cumulative distribution functions: $F^{-1}(u) = \lambda_1 + [u^{\lambda_3} - (1-u)^{\lambda_4}]/\lambda_2$, for $0 \le u \le 1$ and $\lambda_j \in \mathbb{R}$ for $j \in \{1, \ldots, 4\}$. Particular parameters of a generalized lambda family used in Monte Carlo experiments come from Bai and Ng (2005) and can be found in Table 2.

Originally, T+100 observations in each experiment are generated, but first 100 of them are discarded in order to eliminate the effect of the initial observations. The number of repetitions of all experiments is set to $R = 1000$ and the number of observations is set to $T \in \{200, 500, 1000\}$. A particular specification of the quantiles is $p_1 = 0.05$, $p_2 = 0.15$, $p_3 = 0.25$, and $p_4 = 0.35$. The configuration of the sample quantiles reflects: (i) 5 % trimming of extreme values from each tail; and (ii) the number of equally spaced sample quantiles ($k = 4$) is set based on preliminary Monte Carlo experiments.[13] The selection vector is $\boldsymbol{\delta} = (1/4, 1/4, 1/4, 1/4, -2, 1/4, 1/4, 1/4, 1/4)'$.

Table 1: Time series models

M1 model:	$y_t = a_t$
M2 model:	$y_t = 0.5y_{t-1} + a_t$
M3 model:	$y_t = 0.8y_{t-1} - 0.5a_{t-1} + a_t$
H1 model:	$y_t = 1 + 0.5y_{t-1} + \epsilon_t$, $\epsilon_t = a_t\sqrt{h_t}$, $h_t = 0.4 + 0.1\epsilon_{t-1}^2 + 0.5h_{t-1}$
H2 model:	$y_t = 1 + 0.5y_{t-1} + \epsilon_t$, $\epsilon_t = a_t\sqrt{h_t}$, $\log h_t = 0.4 + 0.1a_{t-1}^2 + 0.5\log h_{t-1}$
N1 model:	$y_t = (-2.0 + 0.75y_{t-1} + a_t)I(S_t = 1) + (0.25y_{t-1} + a_t)I(S_t = 2)$
N2 model:	$y_t = (-2.0 + 0.5y_{t-1} + \sqrt{2}a_t)I(S_t = 1) + (0.5y_{t-1} + a_t)I(S_t = 2)$

[13] Note that we leave the issue of optimal number of quantiles for further research.

Table 2: Parameters of a generalized lambda distribution

	λ_1	λ_2	λ_3	λ_4	skewness	kurtosis	moment
S1	0.000000	-1.000000	-0.080000	-0.080000	0.0	6.0	12
S2	0.000000	-0.397912	-0.160000	-0.160000	0.0	11.6	6
A1	0.000000	1.000000	-0.007500	-0.030000	-1.5	7.5	10
A2	0.000000	1.000000	-0.100900	-0.180200	-2.0	21.1	5

[*] Note that a standard normal distribution can also be approximated by a generalized lambda distribution with the following parameters: $\lambda_1 = 0$, $\lambda_2 = 0.1975$, $\lambda_3 = \lambda_4 = 0.1349$.
[a] The highest possible moment satisfied by the distribution.

3.2 MONTE CARLO RESULTS

For each DGP and the configuration of the distribution of model innovations, the average rejection frequency is calculated as follows

$$avg = \frac{1}{R}\sum_{j=1}^{R} I(\hat{\alpha} \leq \alpha), \tag{10}$$

where R denotes the number of repetitions, $I(\cdot)$ is a standard indication function, α is the statistical significance of the test set to 0.05, and $\hat{\alpha}$ is the estimated p-value of the test.

The Monte Carlo results are presented in Tables 5. The results reveal the following: (i) The proposed tests have good size properties, regardless of the sample size (i.e. M1, H1, H2 models with N, S1, S2 innovations). Nevertheless, a small size distortion is observed for more complex ARMA specifications (i.e. M2 and M3 models) with heavy tailed symmetric innovations (i.e. S1 and S2). However, it is important to emphasize that the distortion is not of the magnitude to make the tests unattractive; (ii) The tests have very good power properties, provided that kurtosis does not dominate the stochastic properties of innovations. This fact can be illustrated using, for example, M2 model. In this case, the average rejection frequency of the QS test is 0.98 for A1 configuration of innovations and the sample size $T = 200$. However, once kurtosis dominates the stochastic properties of innovations, a power loss of the tests is observed in small samples. In this case, the average rejection drops from 0.98 to 0.50. However, it is worth pointing out that the power of the test quickly improves as the sample size increases. The average rejection frequency increases from 0.50 in the sample $T = 200$ to 0.88 in the sample $T = 500$; (iii) Significant differences are observed in the behaviour of the tests for two non-linear models (i.e. GARCH and MAR models). As for the GARCH models, asymmetry generated by the functional form of the volatility component is, as might be expected, negligible as compared to the effect of model innovations. In contrast, asymmetry in MAR models can be generated by many different parameter configurations. The results indicate that the tests have very good power for most of the configurations even in small samples, provided that asymmetry is generated by at least two parameters (e.g. regime constants and AR parameters). Of course, the power of the tests significantly improves with asymmetry of innovations. For example, the average rejection frequency of N2 model ranges from 0.86 to 1.00 in the sample $T = 200$, regardless of the distribution of innovations; (iv) Finally, no significant differences in the size and power properties are found between the QS test, a test with the correctly estimated variance-covariance matrix $\mathbf{\Sigma}$, and the QS* test, a test with the approximated variance-covariance matrix $\mathbf{\Sigma}^*$. The fact that the researcher does not have to consider the correct estimation of the variance-covariance matrix and can easily use the approximated one makes the QS* very attractive for applied research.

3.3 Comparison with other tests

The performance of the proposed QS test is compared with two other test statistics: (i) a skewness-based symmetry test developed by Bai and Ng (2005), denoted as NBS; and (ii) a bootstrap-based Kolmogorov-Smirnov test developed by Psaradakis (2003), denoted as BKS. The data generating processes considered for comparison can be found in Table 1. Models M1 and M2 are considered in Bai and Ng (2005), whereas model M3 in Psaradakis (2003). All results are based on $R = 1000$ replications.

Table 3: Comparison of marginal symmetry tests: $T = 200$

	$N(0,1)$		S2		A2	
model	NBS	QS	NBS	QS	NBS	QS
M1	0.05	0.06	0.04	0.06	0.43	0.68
M2	0.04	0.04	0.04	0.07	0.37	0.50
	$N(0,1)$		$t(5)$		$\chi^2(4)$	
model	BKS	QS	BKS	QS	BKS	QS
M3	0.04	0.06	0.04	0.06	0.86	0.90

[a] "BKS" denotes a bootstrap Kolmogorov-Smirnov test discussed in Psaradakis (2003), "NBS" denotes a symmetry test based on a coefficient of skewness discussed in Bai and Ng (2005), "QS" denotes a test for marginal symmetry based on quantiles.
[b] The significance level is set to $\alpha = 0.05$.

The Monte Carlo results are presented in Table 3. The results reveal the following: (i) No differences are noticed in the size properties of all tests (i.e. DGP configurations M1, M2, M3 and symmetric innovations $N(0,1)$, S2, $t(5)$). This means that under the null hypothesis of symmetry, all tests perform similarly; (ii) The power results of the robust QS test are significantly better as compared to the NBS test (i.e. DGP configuration M1, M2 and asymmetric innovations A2), and approximately similar as compared to the BKS test (i.e. DGP configurations M3 with asymmetric innovations $\chi^2(4)$). Our results clearly suggest that, in situations where the user is not proficient in bootstrap or it is not clear which bootstrap method should be implemented, the quantile-based symmetry test may serve as a valuable alternative. Moreover, the computation of the quantile test is considerably easier and faster as compared to any bootstrap test.

4. Empirical application

In this section, the quantile-based symmetry tests QS and QS* (i.e. the correct one and its approximation) are applied to a set of 22 monthly economic time series spanning the period 1980 and 2007.[14] The main task of this exercise is to assess how consistent the results obtained from both quantile symmetry tests are in practice. That means whether both tests lead to the same conclusion about rejecting and/or not rejecting the null hypothesis of symmetry. Moreover, we are interested in answering the question whether the degree of asymmetry is similar across different asset classes or not. This issue is of much practical importance in finance for both asset pricing and/or risk management when constructing and evaluating financial portfolios.

The empirical results (the estimated p-values of the QS and QS* tests) are presented in Table 4. The results suggest the following: (i) Both tests lead to the same conclusion (i.e. rejecting or not rejecting the null of symmetry) in 21 out of 22 cases at the significance level 0.10. This finding fully supports the Monte Carlo results that the QS* test, based on the approximated variance-covariance matrix $\mathbf{\Sigma}^*$, produces almost identical results as compared to the QS test, based on the correctly estimated variance-covariance matrix $\mathbf{\Sigma}$; (ii) The null hypothesis of symmetry is rejected in 8 of 22 cases by the QS test (i.e. in 36 $\%$). However, noticeable differences are observed among various classes of financial assets. For example, the null hypothesis is rejected for 3 out of 5 exchange rate returns (i.e. in 60 $\%$), whereas for just 1 out of 6 commodities (i.e. in 17 $\%$). Much more importantly, the difference in the rejection frequencies of exchange rate and commodity returns is statistically significant at the nominal level 0.10.[15] Therefore, it can be concluded that the degree of asymmetry varies across asset classes.

5. Conclusion and further research

A modified quantile-based symmetry test of the marginal law of weakly dependent stochastic processes has been proposed in this article. It has been shown that the test is intuitive, easy to calculate, follows standard limiting distribution, and much more importantly, it is robust against outliers and weak dependence of observations. Especially the last feature makes the test very attractive for applied research in economics and/or finance since it reduces the inferential errors coming from the incorrect estimation of the key quantities of the test. Monte Carlo results

[14] This time period is used on purpose. We want to eliminate the effect of oil price shocks in 1970's and the last financial crises after 2007.

[15] Any other differences in rejection frequencies are not statistically significant at the nominal level 0.10.

Table 4: Testing marginal symmetry: period 1980M1 – 2007M12

variable	QS	QS*	variable	QS	QS*
Exchange rates			**Interest rates**		
USDGBP	0.64	0.65	USIR3M	0.07	0.02
USDJPY	0.01	0.01	UKIR3M	0.60	0.51
USDCAD	0.59	0.52	CAIR3M	0.09	0.07
USDAUD	0.09	0.08	AUIR3M	0.17	0.08
USDCHF	0.07	0.06	CHIR3M	0.94	0.92
Equities			**Commodities**		
DJIA	0.58	0.56	WHEAT	0.55	0.55
TOPIX	0.25	0.27	SOYBN	0.79	0.80
FTUK	0.81	0.79	COFFEE	0.40	0.37
TSE	0.01	0.01	COTTON	0.94	0.93
AUSE	0.42	0.39	FUEL	0.86	0.86
CHSE	0.00	0.01	GOLD	0.08	0.10

[a] Monthly averages of daily observations.

[b] Note that a particular transformation of each series is indicated in the brackets. USDGBP = the US dollar to British pound exchange rate ($\Delta \log$), USDJPY = the US dollar to Japanese yen exchange rate ($\Delta \log$), USDCAD = the US dollar to Canadian dollar exchange rate ($\Delta \log$), USDAUD = the US dollar to Australian dollar exchange rate ($\Delta \log$), USDCHF = the US dollar to Swiss frank exchange rate ($\Delta \log$), DJIA = the US Dow Jones Industrials Share Index ($\Delta \log$), UKFT = the UK FT All Shares Index ($\Delta \log$), TOPIX = Tokyo Stock Exchange Index ($\Delta \log$), TSE = the Toronto Stock Exchange Index ($\Delta \log$), AUSE = Australian Stock Exchange Index ($\Delta \log$), CHSE = the Swiss Stock Exchange Index ($\Delta \log$), WHEAT = Kansas wheat, hard, cents/bushel ($\Delta \log$), SOYBEAN = soybeans, yellow, cents/bushel ($\Delta \log$), COFFEE = Brazilian coffee beans cents/pound ($\Delta \log$), COTTON = cotton, cents/pound ($\Delta \log$), FUEL = fuel oil, cents/gallon ($\Delta \log$), GOLD = gold bullion, USD/troy ounce ($\Delta \log$), USIR = the US interest rates, 3M (Δ), UKIR = the UK interest rates, 3M (Δ), CAIR = the Canadian interest rates, 3M (Δ), AUIR = the Australian interest rates, 3M (Δ), CHIR = the Swiss interest rates, 3M (Δ).

suggest that the finite sample properties of the QS test significantly outperforms the skewness-based symmetry test and compares favorably with the bootstrap-based Kolmogorov-Smirnov test. It can be concluded that in situations where the user is not proficient in bootstrap, or it is not clear which bootstrap method should be implemented, the QS test may serve as a valuable alternative.

One possible caveat of this paper is that the configuration of sample quantiles is to same extent arbitrary. Therefore, a natural extension of the paper is to focus on "optimal" determination of the number and location of sample quantiles. This interesting issue is subject to ongoing research and the results will be published in a separate article.

REFERENCES

Acemoglu, D. and A. Scott (1997). Asymmetric business cycles: Theory and time-series evidence. *Journal of Monetary Economics 40*(3), 501–533.

Andrews, D. (1991). Heteroskedasticity and autocorrelation consistent covariance matrix estimation. *Econometrica 59*(3), 817–858.

Bai, J. and S. Ng (2005). Tests for skewness, kurtosis, and normality for time series data. *Journal of Business and Economic Statistics 23*(1), 49–60.

Bali, T. (2003). An extreme value approach to estimating volatility and value at risk. *The Journal of Business 76*(1), 83–108.

Balke, N. and T. Fomby (1994). Large shocks, small shocks, and economic fluctuations: outliers in macroeconomic time series. *Journal of Applied Econometrics 9*(2), 181–200.

Beirlant, J. (2004). *Statistics of Extremes: theory and applications*. Wiley.

Black, F. and M. Scholes (1973). The pricing of options and corporate liabilities. *The journal of Political Economy*, 637–654.

Boldin, M. (1999). Should policy makers worry about asymmetries in the business cycle? *Studies in Nonlinear Dynamics and Econometrics 3*(4), 203–220.

Bowman, K. and L. Shenton (1975). Tables of moments of the skewness and kurtosis statistics in non-normal sampling. *Union Carbide Nuclear Divison Report*.

Cai, Z. and G. G. Roussas (1992). Uniform strong estimation under α-mixing, with rates. *Statistics & probability letters 15*(1), 47–55.

Chanda, K. (1974). Strong mixing properties of linear stochastic processes. *Journal of Applied Probability 11*(2), 401–408.

Corrado, C. and T. Su (1996). Skewness and kurtosis in S&P 500 index returns implied by option prices. *Journal of Financial research 19*, 175–192.

De Jong, R. (2000). A strong consistency proof for heteroskedasticity and autocorrelation consistent covariance matrix estimators. *Econometric Theory 16*, 262–268.

Dehling, H. and W. Philipp (2002). Empirical process techniques for dependent data. In H. Dehling, T. Mikosch, and M. Sørensen (Eds.), *Empirical Process Techniques for Dependent Data*. Springer.

Doukhan, P. (1994). *Mixing: properties and examples*. Springer.

Francq, C. and J. Zakoïan (2006). Mixing properties of a general class of GARCH(1,1) models without moment assumptions on the observed process. *Econometric Theory 22*(05), 815–834.

Groeneveld, R. and G. Meeden (1984). Measuring skewness and kurtosis. *The Statistician 33*(4), 391–399.

Hall, P., S. Lahiri, and Y. Truong (1995). On bandwidth choice for density estimation with dependent data. *The Annals of Statistics 23*(6), 2241–2263.

Hansen, B. (1992). Consistent covariance matrix estimation for dependent heterogeneous processes. *Econometrica 60*(4), 967–972.

Hansen, B. E. (2008). Uniform convergence rates for kernel estimation with dependent data. *Econometric Theory 24*(3), 726.

Hinkley, D. (1975). On power transformations to symmetry. *Biometrika 62*(1), 101–111.

Ibragimov, I. and Y. Linnik (1971). Independent and stationary sequences of random variables.

Jarrow, R. and A. Rudd (1982). Approximate option valuation for arbitrary stochastic processes. *Journal of Financial Economics 10*(3), 347–369.

Jones, M., J. Marron, and S. Sheather (1996). A brief survey of bandwidth selection for density estimation. *Journal of the American Statistical Association 91*(433), 401–407.

Koedijk, K., M. Schafgans, and C. de Vries (1990). The tail index of exchange rate returns. *Journal of International Economics 29*(1-2), 93–108.

Lahiri, S. and S. Sun (2009). A Berry–Esseen theorem for sample quantiles under weak dependence. *The Annals of Applied Probability 19*(1), 108–126.

Lehmann, E. (1999). *Elements of Large-sample Theory*. Springer.

Liebscher, E. (1996). Strong convergence of sums of α-mixing random variables with applications to density estimation. *Stochastic Processes and their Applications 65*(1), 69–80.

Liebscher, E. (2005). Towards a unified approach for proving geometric ergodicity and mixing properties of nonlinear autoregressive processes. *Journal of Time Series Analysis 26*(5), 669–689.

Newey, W. and D. Steigerwald (1997). Asymptotic bias for quasi-maximum-likelihood estimators in conditional heteroskedasticity models. *Econometrica 65*(3), 587–599.

Newey, W. and K. West (1994). Automatic lag selection in covariance matrix estimation. *The Review of Economic Studies 61*(4), 631.

Peiró, A. (1999). Skewness in financial returns. *Journal of Banking & Finance 23*(6), 847–862.

Pope, A. (1990). Biases of estimators in multivariate non-gaussian autoregressions. *Journal of Time Series Analysis 11*(3), 249–258.

Premaratne, G. and A. Bera (2005). A test for symmetry with leptokurtic financial data. *Journal of Financial Econometrics 3*(2), 169.

Psaradakis, Z. (2003). A bootstrap test for symmetry of dependent data based on a Kolmogorov–Smirnov type statistic. *Communications in Statistics 32*(1), 113–126.

Randles, R., M. Fligner, G. Policello, and D. Wolfe (1980). An asymptotically distribution-free test for symmetry versus asymmetry. *Journal of the American Statistical Association 75*(369), 168–172.

Runde, R. (1997). The asymptotic null distribution of the Box-Pierce Q-statistic for random variables with infinite variance an application to German stock returns. *Journal of Econometrics 78*(2), 205–216.

Schott, J. (2005). *Matrix Analysis for Statistics*. Wiley.

Serfling, R. (1980). *Approximation Theorems of Mathematical Statistics*. Wiley.

Sharipov, O. and M. Wendler (2013). Normal limits, nonnormal limits, and the bootstrap for quantiles of dependent data. *Statistics & Probability Letters 83*, 1028–1035.

Silverman, B. (1986). *Density Estimation for Statistics and Data Analysis*. Chapman & Hall/CRC.

Sims, C. (1980). Macroeconomics and reality. *Econometrica 48*(1), 1–48.

Sun, S. and S. Lahiri (2006). Bootstrapping the sample quantile of a weakly dependent sequence. *Sankhyā: The Indian Journal of Statistics 68*(1), 130–166.

Tiku, M., W. Wong, and G. Bian (2001). Estimating parameters in autoregressive models in non-normal situations: asymmetric innovations. *Communications in Statistics 28*(2), 315–341.

Vahamaa, S. (2003). Skewness and kurtosis adjusted black-scholes model: a note on hedging performance. *Finance Letters 1*(5), 6–12.

White, H. (2001). *Asymptotic Theory for Econometricians*. Academic Press.

Yang, M. and R. Bewley (1995). Moving average conditional heteroskedastic processes. *Economics Letters 49*(4), 367–372.

A. PROOFS

It is assumed that all conditions in Assumptions 1 – 3 are implicitly satisfied.

A.1 PROOF OF THEOREM 1

Since $\xi_p \in \boldsymbol{\xi}$ for any $p \in \mathcal{P}_k$, it is fully sufficient to show that $\hat{\xi}_p - \xi_p \xrightarrow{a.s.} 0$ as $T \to \infty$. Noting that the process $\{Y_t\}$ is ergodic when $\lim \alpha(n) = 0$ as $n \to \infty$, see White (2001, Proposition 3.44), we have $\hat{F}(y) - F(y) \xrightarrow{a.s.} 0$ as $T \to \infty$, uniformly in $y \in \mathbb{R}$, on account of Glivenko-Cantelli theorem for stationary and ergodic processes (e.g. Dehling and Philipp (2002, Theorem 1.1)). The assertion then follows by a standard argument about mapping between a distribution function F and a quantile ξ_p (e.g. Serfling (1980, p. 75)).

A.2 PROOF OF THEOREM 2

Under the conditions of the theorem, for every $p \in \mathcal{P}_k$, ξ_p admits the following Bahadur representation

$$\hat{\xi}_p - \xi_p = \frac{p - \hat{\mathbb{F}}(\xi_p)}{f(\xi_p)} + R = \frac{1}{T}\sum_{t=1}^{T}\left(\frac{p - I(Y_t \leq \xi_p)}{f\xi_p)}\right) + R, \tag{11}$$

where $R = o(1/\sqrt{T})$ as $T \to \infty$, see Theorem 1 in Sharipov and Wendler (2013). Hence, we have

$$\sqrt{T}(\hat{\boldsymbol{\xi}} - \boldsymbol{\xi}) = \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathbf{Z}_t + o_p(\mathbf{1}), \tag{12}$$

where

$$\mathbf{Z}_t = \left(\frac{p_1 - I\left(Y_t \leq \xi_{p_1}\right)}{f(\xi_{p_1})}, \ldots, \frac{1 - p_1 - I\left(Y_t \leq \xi_{1-p_1}\right)}{f(\xi_{1-p_1})}\right)'$$

, so it remains to establish asymptotic normality of $\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathbf{Z}_t$. By the properties of $\{Yt\}$, for any $p \in \mathcal{P}_k$, $p - I(Y_t \leq \xi_p)$ is a strictly stationary sequence of bounded random variables with α-mixing coefficients of the same size as $\alpha(n)$, see Theorem 3.49 in White (2001), and $\mathbb{E}(p - I(Y_t \leq \xi_p)) = p - F(\xi_p) = 0$ on account of the assumed continuity of F at ξ_p. Thus, by an application of the central limit theorem for stationary α-mixing, see Theorem 18.5.4 in Ibragimov and Linnik (1971), we may conclude that $\frac{1}{\sqrt{T}}\sum_{t=1}^{T}[p - I(Y_t \leq \xi_p)]/f(\xi_p)$ has a limiting normal distribution with mean zero and variance $f^{-2}(\xi_p)\sum_{h=-\infty}^{\infty}\mathrm{cov}[I(Y_1 \leq \xi_p), I(Y_{1+h} \leq \xi_p)]$. By considering arbitrary linear combination of the components of $\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathbf{Z}_t$ and applying the central limit theorem the required result follows via the Cramér-Wold device.

A.3 PROOF OF COROLLARY 1

It follows from Theorem 2 that $\sqrt{T}(\hat{\boldsymbol{\xi}} - \boldsymbol{\xi}) \xrightarrow{d} N(\mathbf{0}, \boldsymbol{\Sigma})$, where $\hat{\boldsymbol{\xi}}$ is a $(2k+1 \times 1)$ vector of sample quantiles and $\boldsymbol{\Sigma}$ is a positive definite variance-covariance matrix. It can be shown that there exists a lower triangular matrix $\mathbf{P}$ such that $\boldsymbol{\Sigma} = \mathbf{P}\mathbf{P}'$, see Theorem 4.3 in Schott (2005, p. 139). Then, the standardized vector $\hat{\mathbf{z}} = \mathbf{P}^{-1}(\hat{\boldsymbol{\xi}} - \boldsymbol{\xi})$ is distributed as $\sqrt{T}\hat{\mathbf{z}} \xrightarrow{d} N(\mathbf{0}, \mathbf{I})$. Using the standardized vector of quantiles $\hat{\mathbf{z}}$, the QS test statistic can be formally written in the following quadratic form

$$QS = T\left(\hat{\mathbf{z}}'\mathbf{A}\hat{\mathbf{z}}\right), \qquad (13)$$

where $\mathbf{A} = \boldsymbol{\delta}\boldsymbol{\delta}'$. Then the limiting $\chi^2(1)$ distribution immediately follows from Theorem 9.8 in Schott (2005, p. 378) about the limiting distribution of a quadratic form of standard normal random variables. The degrees of freedom follow from the fact that $\mathbf{A}$ is a matrix with $\mathrm{rk}(\mathbf{A}) = 1$.

B. Tables

Table 5: Statistical properties of the DGP processes

		T=200		T=500		T=1000	
DGP	distr.	QS	QS*	QS	QS*	QS	QS*
M1	N	0.06	0.05	0.05	0.05	0.05	0.05
	S1	0.05	0.05	0.05	0.05	0.05	0.05
	S2	0.06	0.05	0.07	0.07	0.05	0.05
	A1	0.99	0.99	1.00	1.00	1.00	1.00
	A2	0.68	0.68	0.98	0.98	1.00	1.00
M2	N	0.04	0.05	0.06	0.07	0.05	0.06
	S1	0.07	0.08	0.05	0.07	0.06	0.08
	S2	0.07	0.09	0.07	0.09	0.06	0.09
	A1	0.98	0.99	1.00	1.00	1.00	1.00
	A2	0.50	0.56	0.88	0.92	0.99	1.00
M3	N	0.06	0.06	0.05	0.06	0.05	0.06
	S1	0.06	0.07	0.06	0.07	0.06	0.08
	S2	0.06	0.07	0.06	0.07	0.06	0.08
	A1	0.98	0.98	1.00	1.00	1.00	1.00
	A2	0.51	0.53	0.86	0.89	0.99	1.00
H1	N	0.06	0.06	0.06	0.06	0.04	0.05
	S1	0.06	0.06	0.05	0.08	0.05	0.07
	S2	0.08	0.09	0.05	0.08	0.05	0.08
	A1	0.97	0.98	1.00	1.00	1.00	1.00
	A2	0.50	0.49	0.86	0.86	0.99	0.99
H2	N	0.06	0.07	0.04	0.05	0.06	0.07
	S1	0.06	0.07	0.06	0.07	0.05	0.08
	S2	0.08	0.10	0.06	0.07	0.07	0.09
	A1	0.98	0.99	1.00	1.00	1.00	1.00
	A2	0.52	0.52	0.86	0.85	0.99	0.99
N1	N	0.86	0.86	1.00	1.00	1.00	1.00
	S1	0.95	0.96	1.00	1.00	1.00	1.00
	S2	0.97	0.98	1.00	1.00	1.00	1.00
	A1	1.00	1.00	1.00	1.00	1.00	1.00
	A2	1.00	1.00	1.00	1.00	1.00	1.00
N2	N	0.42	0.45	0.79	0.81	0.98	0.98
	S1	0.44	0.47	0.81	0.83	0.98	0.98
	S2	0.45	0.47	0.84	0.86	0.99	0.99
	A1	0.88	0.89	1.00	1.00	1.00	1.00
	A2	0.73	0.75	0.99	0.99	1.00	1.00

[a] QS denotes a marginal symmetry test with the correct variance-covariance matrix $\mathbf{\Sigma}$, whereas QS* denotes a marginal symmetry test with the approximated variance-covariance matrix $\mathbf{\Sigma}^*$.