NBS Working paper
10/2026

Bootstrap-Assisted Tests for Skewness, Kurtosis, and Normality Under Unknown Dependence

Charisios Grivas, Zacharias Psradakis, Marián Vávra

NATIONAL BANK OF SLOVAKIA

www.nbs.sk

The publication has not undergone language editing.
Without the authors' prior consent, only brief excerpts, no more than two paragraphs, may be used, provided the source is cited.

The views and results presented in this paper are those of the authors and do not necessarily represent the official opinion of the National Bank of Slovakia.

Publisher
© National Bank of Slovakia 2026
research@nbs.sk

Contact
National Bank of Slovakia
Imricha Karvaša 1
813 25 Bratislava

Electronic version
https://nbs.sk/en/publications/research-papers-working-and-occasional-papers-wp-op/



ISSN 2585-9269 (electronic version)

Bootstrap-Assisted Tests for Skewness, Kurtosis, and Normality Under Unknown Dependence*

Charisios Grivas† Zacharias Psaradakis‡ Marián Vávra§

November 4, 2026

Abstract

This paper considers tests for skewness, kurtosis and normality of the one-dimensional marginal distribution of a stationary fractional process, which may exhibit short memory, antipersistence or long memory. The autoregressive-sieve bootstrap is used to obtain P-values and critical values for tests based on studentized empirical coefficients of skewness and kurtosis. These studentized coefficients are constructed using variance estimators obtained by the same bootstrap procedure. Notably, such tests, as well as variants based on non-studentized statistics, can be implemented in a straightforward manner without requiring knowledge or estimation of the memory parameter of the process. Monte Carlo simulations show that the bootstrap-assisted tests perform well for sample sizes that are typical in applied settings. The methodology is further illustrated through an application to realized measures of stock-return volatility.

JEL code: C12, C22, C63.
Keywords: Autoregressive-sieve bootstrap; Fractional process; Kurtosis; Normality; Realized volatility; Skewness.

*The authors are solely responsible for the content of the paper. The views expressed are solely those of the authors and can under no circumstances be regarded as representing an official position of the National Bank of Slovakia or the Eurosystem. We are solely responsible for any remaining errors.

†Department of Mathematical Sciences, Aalborg University, Fibigerstræde 2, 9220 Aalborg, Denmark (cgrivas@math.aau.dk).

‡Birkbeck Business School, Birkbeck, University of London, Malet Street, London WC1E 7HX, United Kingdom (z.psaradakis@bbk.ac.uk).

§Corresponding author. National Bank of Slovakia, Imricha Karvaša 1, 81325 Bratislava, Slovak Republic; Institute for Forecasting, Slovak Academy of Sciences, Šancová 56, 81105 Bratislava, Slovak Republic. (marian.vavra@nbs.sk).

1. Introduction

The problems of testing for symmetry, mesokurtosis or Gaussianity of a probability distribution have attracted considerable attention. This is perhaps unsurprising, given that normality or symmetry is commonly assumed in a wide range of statistical inference and prediction procedures. In model building exercises, tests for symmetry or normality are useful specification tests, since deviations from symmetry or normality in the marginal distribution of the data offer justification for excluding specific families of parametric models as possible data-generating processes. For example, asymmetry (resp. non-normality) of the one-dimensional marginal distribution of a strictly stationary stochastic process rules out linear autoregressive integrated moving average (ARIMA) models, or nonlinear Markovian models with skew-symmetric autoregressive functions, whose driving noise is independently and symmetrically (resp. normally) distributed. Symmetry and normality are also important considerations in analyzing time-reversibility of a stochastic process, that is, invariance of ensemble properties with respect to the direction of time. Specifically, symmetry of the distribution of first- and higher-order differences of a process is necessary for time-reversibility, while for linear processes, normality of marginal distributions is typically equivalent to time-reversibility (Cheng (1999)).

Tests for symmetry or normality can also be helpful in assessing the validity of theoretical models and hypotheses when those rely on or imply such characteristics, as is the case with many pricing models, rational expectations models and dynamic stochastic general equilibrium models found in economics and finance. A prominent example from economics where symmetry is a central issue is the long-standing debate over whether real economic variables exhibit asymmetric behavior over the phases of a business cycle. Various forms of cyclical asymmetry can be identified and detected via either distributional asymmetry or time-irreversibility of relevant economic and financial time series (cf. Sichel (1993), Ramsey and Rothman (1996)). In finance, symmetry and/or normality is an implicit or explicit assumption in several widely used models, including the Black–Scholes option-pricing model, the Sharpe–Lintner asset-pricing model, and models with elliptically distributed multivariate asset returns, on which much of classical mean-variance analysis of portfolio choice and equilibrium asset pricing is based (see, e.g., Ingersoll (1987)). Moreover, tail-heaviness and normality play a central role in value-at-risk calculations and related risk-management problems (see, e.g., Gourieroux and Jasiak (2010)), as

well as in copula-based modelling of multivariate time series, where the marginal distributions and the copula function are specified separately. Kilian and Demiroglu (2000), Bontemps and Meddahi (2005) and Psaradakis and Vávra (2022) provide further examples from economics, finance, econometrics and statistics where symmetry and/or normality are of interest.

While the majority of the extensive research on tests for symmetry, mesokurtosis and normality has primarily focused on the case of independent and identically distributed (i.i.d.) data, several studies have explored approaches that remain valid when serial correlation is present; see Psaradakis and Vávra (2019, 2020) and the references therein. A common assumption in most of these studies is that observations constitute a realization of a stochastic process whose autocovariance sequence decays towards zero sufficiently fast to be absolutely summable; such processes are said to have short memory, or short-range dependence. However, it has been well established across fields as varied as economics, geophysics, hydrology, meteorology and telecommunications that many data sets are characterized by slowly decaying correlations, suitable models for which are provided by stochastic processes whose autocovariance sequences are not absolutely summable; such processes are said to have long memory, or long-range dependence.

In this paper, we consider the problem of testing for symmetry, mesokurtosis and normality of the one-dimensional marginal distribution of a strictly stationary stochastic process whose memory/dependence structure is unspecified. Like Lobato and Velasco (2004), Bai and Ng (2005) and Peng-Zhou and Song (2024), we focus on tests based on classical measures of skewness and kurtosis involving empirical standardized third-order and fourth-order central moments. Such tests are known to have high power under general conditions, are easy to implement (particularly in i.i.d. settings) and are extensively used in practice. Unlike the aforementioned studies, however, which rely on short-memory assumptions, we consider fractional, or fractionally integrated, processes that may exhibit long memory, short memory or antipersistence (autocovariance sequence is absolutely summable with sum zero), depending on the value of their memory, or fractional differencing, parameter.

While our setup affords considerable flexibility and accommodates types of serial dependence more general than those considered elsewhere in the literature, statistical inference based on the empirical coefficients of skewness and kurtosis within this setting is challenging. The difficulty stems primarily from the fact that the asymptotic distributions of the moment-based

measures of skewness and kurtosis, and associated rates of convergence, differ across values of the (unknown) memory parameter; see, inter alia, Beran and Ghosh (1991), Ho (2002) and Boutahar (2010). Besides the evident problem of identifying the appropriate asymptotic distribution for a given data set, large-sample inference is further complicated by the potential non-Gaussianity of the asymptotic distributions and the difficulty of estimating their variances with high precision.

To address these challenges, we propose using a suitable bootstrap procedure to: (a) estimate the variances of the empirical coefficients of skewness and kurtosis; (b) approximate the finite-sample distributions of the resulting studentized coefficients of skewness and kurtosis (or, alternatively, those of the non-studentized coefficients) under appropriate assumptions about the underlying distribution of the data. Our approach relies on the autoregressive-sieve bootstrap (ARSB), introduced in short-memory settings by Kreiss (1992) and Bühlmann (1997), and extended to the case of fractional processes by Kapetanios and Psaradakis (2006) and Poskitt (2008). This method is based on the idea of approximating the data-generating process by an autoregressive sieve, that is, a sequence of autoregressive models whose order increases without bound at an appropriate rate as the sample size tends to infinity. In our setting, the ARSB delivers tests for symmetry, mesokurtosis and normality which are straightforward to implement, asymptotically correct and require no knowledge or estimation of the memory parameter of the data or the appropriate scaling factor for the test statistic. Moreover, the bootstrap procedure remains the same for antipersistent, short-memory and long-memory processes. This allows for implementation without the need for prior assumptions or pre-testing relating to the memory parameter, which further enhances the attractiveness of the method.

ARSB-assisted tests for symmetry or normality of the one-dimensional marginal distribution of fractional processes were previously considered by Psaradakis (2016) and Psaradakis and Vávra (2017). The former explored tests for symmetry based on linear statistics that are odd functions of centred data, including the third-degree monomial function associated with the familiar moment-based measure of skewness. The latter examined a test for normality based on the (weighted) quadratic distance of the empirical distribution function from a Gaussian distribution function. Boutahar (2010) obtained results relating to the asymptotic behavior of the empirical coefficients of skewness and kurtosis for a class of Gaussian, long-memory processes, but did not discuss how operational tests for symmetry, mesokurtosis and normality

might be constructed. To the best of our knowledge, this paper is the first attempt to develop such tests within a unified framework that encompasses multiple types of dependence.

The remainder of the paper is organized as follows. Section 2 lays out the basic framework, data-generating process and tests of interest. Section 3 discusses how the ARSB can be used to implement the tests and justifies the approach asymptotically. Section 4 presents the results of a simulation study of the finite-sample properties of ARSB-assisted tests. Section 5 provides an empirical illustration using time series of realized measures of equity-return volatility. Section 6 summarizes and concludes. The Supplementary Material provides a complete set of simulation results.

2. Framework, Test Statistics and Assumptions

Consider a strictly stationary, real-valued stochastic process $\mathbf{X} := \{X_t\}_{t\in\mathbb{Z}}$ having mean $\mu := \mathsf{E}(X_0)$ and autocovariance sequence $\gamma_k := \mathrm{Cov}(X_k, X_0)$, $k \in \mathbb{Z}$, with $\gamma_0 > 0$ ($\mathbb{Z}$ denotes the set of all integers). Provided X_0 has finite fourth moment, hypotheses about symmetry, tail-heaviness/peakedness and Gaussianity of the one-dimensional marginal distribution of $\mathbf{X}$ may be formulated in terms of the coefficients of skewness and kurtosis of X_0, defined as $\beta_1 := \mathsf{E}(\tilde{X}_0^3)$ and $\beta_2 := \mathsf{E}(\tilde{X}_0^4)$, respectively, where $\tilde{X}_t := \gamma_0^{-1/2}(X_t - \mu)$. Symmetry of the distribution of X_0 about μ implies that $\beta_1 = 0$, while $\beta_2 = 3$ indicates that the distribution of X_0 is as dispersed around the points $\mu \pm \gamma_0^{1/2}$ as a Gaussian distribution.

Given a finite segment $\mathbf{X}^n := \{X_t\}_{t=1}^n$ of $\mathbf{X}$, natural (two-sided) tests of the hypotheses $\beta_1 = 0$ and $\beta_2 = 3$ reject for large absolute values of the statistics

$$S_n := \hat{\beta}_{1,n}/\hat{\omega}_{1,n}, \tag{1}$$

and

$$K_n := (\hat{\beta}_{2,n} - 3)/\hat{\omega}_{2,n}, \tag{2}$$

respectively. Here, $\hat{\beta}_{1,n}$ and $\hat{\beta}_{2,n}$ are the empirical coefficients of skewness and kurtosis, defined as

$$\hat{\beta}_{r,n} := n^{-1}\sum_{t=1}^{n}[\hat{\gamma}_{0,n}^{-1/2}(X_t - \bar{X}_n)]^{r+2}, \quad r \in \{1,2\},$$

with $\bar{X}_n := n^{-1}\sum_{t=1}^n X_t$ and $\hat{\gamma}_{0,n} := n^{-1}\sum_{t=1}^n (X_t - \bar{X}_n)^2$, and $\hat{\omega}_{r,n}$ is a consistent estimator of the (asymptotic) standard deviation of $\hat{\beta}_{r,n}$. The hypothesis $(\beta_1, \beta_2) = (0, 3)$ associated with

normality of X_0 may be rejected for large values of the composite statistic

$$G_n := S_n^2 + K_n^2, \tag{3}$$

as suggested by Bowman and Shenton (1975) and Jarque and Bera (1987). In an i.i.d. setting, a test based on G_n is known to be a score test for normality within the Pearson family of distributions or within the family of distributions defined by a truncated, three-term Gram–Charlier series (Kiefer and Salmon (1983), Jarque and Bera (1987)). A test based on S_n has optimal power properties for testing symmetry of i.i.d. data against alternatives that belong to certain families of skewed distributions that are in the vicinity of normality (Cassart et al. (2011)).

When $\mathbf{X}$ consists of i.i.d. Gaussian variables, S_n and K_n are typically constructed with $\hat{\omega}_{1,n}^2 = 6/n$ and $\hat{\omega}_{2,n}^2 = 24/n$. In this case, the resulting statistics are asymptotically standard normal (as $n \to \infty$), while the composite statistic G_n is asymptotically chi-squared with two degrees of freedom. For dependent data, Lomnicki (1961), Gasser (1975) and Bai and Ng (2005) obtained the asymptotic variances of $\hat{\beta}_{1,n}$ and $\hat{\beta}_{2,n}$ under appropriate dependence and moment conditions, while Lobato and Velasco (2004) and Bai and Ng (2005) discussed how to estimate such quantities consistently. More recently, Peng-Zhou and Song (2024) considered the asymptotic properties of self-normalized variants of $\hat{\beta}_{1,n}$ and $\hat{\beta}_{2,n}$. A common assumption in all of these studies is that $\mathbf{X}$ satisfies the short-memory condition $\sum_{k\in\mathbb{Z}} |\gamma_k| < \infty$, under which $\hat{\beta}_{1,n}$ and $\hat{\beta}_{2,n}$ are asymptotically normal (with appropriate centering and $\sqrt{n}$ scaling).

The situation changes dramatically if the autocovariance sequence of $\mathbf{X}$ is not summable. For Gaussian, long-memory, linear processes, Boutahar (2010) showed that the behavior of $\hat{\beta}_{1,n}$ and $\hat{\beta}_{2,n}$, and composite statistics combining $\hat{\beta}_{1,n}^2$ and $(\hat{\beta}_{2,n} - 3)^2$, depends critically on the value of the memory parameter controlling the rate of asymptotic decay of the autocovariances. Depending on the value of this parameter, the statistics in question may have asymptotic distributions that are not normal or chi-squared, with rates of convergence that are often slower than $\sqrt{n}$ or n.

To accommodate both short and long memory in our subsequent analysis, it is maintained throughout that $\mathbf{X}$ satisfies the equations

$$X_t - \mu = (1 - L)^{-d}\,\theta(L)\varepsilon_t, \qquad t \in \mathbb{Z}, \tag{4}$$

for some $d \in (-1/2, 1/2)$ and $\theta(L) := \sum_{j=0}^{\infty} \theta_j L^j$, $\theta_0 := 1$, where L denotes the lag operator ($L\varepsilon_t = \varepsilon_{t-1}$) and the following hold:

C1: $\{\theta_j\}_{j=1}^{\infty}$ are real coefficients such that $\sum_{j=0}^{\infty} |\theta_j| < \infty$ and $|\theta(z)| \geq C_\theta > 0$ for all complex z with $|z| \leq 1$;

C2: $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ are i.i.d., real random variables such that $\mathsf{E}(\varepsilon_0) = 0$, $\mathsf{E}(\varepsilon_0^2) = \sigma^2 > 0$, and $\mathsf{E}(|\varepsilon_0|^{4\nu}) < \infty$ for some $\nu \geq 1$.

As usual, the operator $(1-L)^{-d}$ is defined as $(1-L)^{-d} := \sum_{j=0}^{\infty} \delta_j L^j$, where $\delta_0 = 1$, $\delta_j = [\Gamma(j+1)\Gamma(d)]^{-1}\Gamma(j+d) = j^{-1}(j-1+d)\delta_{j-1}$, $j \geq 1$, and $\Gamma(\cdot)$ is the gamma function (with the convention $1/\Gamma(0) = 0$).

The stochastic process defined by (4) is a fractional process with memory parameter d, which admits the causal and invertible moving-average representation

$$X_t - \mu = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}, \qquad t \in \mathbb{Z}, \tag{5}$$

with $\psi_j = \sum_{i=0}^{j} \delta_i \theta_{j-i}$ and $\sum_{j=0}^{\infty} \psi_j^2 < \infty$. Hence, for $d \neq 0$, $\psi_j \sim [\theta(1)/\Gamma(d)]j^{d-1}$ as $j \to \infty$ and $\gamma_k \sim c_d |k|^{2d-1}$ as $|k| \to \infty$, with $c_d := \pi^{-1}[\sigma\theta(1)]^2\Gamma(1-2d)\sin(\pi d)$ (the symbol $\sim$ signifies that the ratio of the two sides approaches 1 in the limit). Therefore, $\sum_{k\in\mathbb{Z}} |\gamma_k| = \infty$ for $d \in (0, 1/2)$ and $\mathbf{X}$ has long memory. If $d = 0$, then $\sum_{k\in\mathbb{Z}} \gamma_k = [\sigma\theta(1)]^2$ and $\mathbf{X}$ has short memory, whereas $\sum_{k\in\mathbb{Z}} \gamma_k = 0$ for $d \in (-1/2, 0)$ and $\mathbf{X}$ is antipersistent.[1]

The class of fractional processes defined above is rich enough to include Gaussian and non-Gaussian linear processes having autocovariance sequences that decay at various rates and spectral densities that have a pole, vanish, or are continuous and positive at the origin. A well-known special case are causal and invertible fractional ARIMA (FARIMA) processes, for which $\theta(\cdot)$ is the ratio of two relatively prime polynomials of finite degree whose zeros have modulus greater than 1. We note that for a linear process satisfying (4), symmetry (resp. normality) of the distribution of ε_0 implies symmetry (resp. normality) of the distribution of X_0. Conversely, normality of X_0 implies normality of ε_0, which in turn implies Gaussianity of $\mathbf{X}$. Hence, a test based on the composite statistic in (3) may also be viewed in our setting as a test for Gaussianity of the fractional process $\mathbf{X}$. For such processes, there is no loss of generality involved

[1] Note that β_1 and β_2 are well-defined under the stated conditions on $\mathbf{X}$, since, by the Marcinkiewicz–Zygmund inequality (see Chow and Teicher (1997, p. 386)), $\mathsf{E}[(X_0 - \mu)^4] \leq C_0(\sum_{j=0}^{\infty} \psi_j^2)^2 \mathsf{E}(\varepsilon_0^4) < \infty$ for some real constant $C_0 > 0$.

in assuming that they are linear in the strict sense, that is, they admit a linear representation with respect to i.i.d. innovations—the existence of such a representation is guaranteed by the Wold decomposition theorem. Linearity in the weak sense, that is, linearity with respect to uncorrelated but not necessarily independent innovations, is not well-suited to the type of statistical problems considered here and is ruled out by condition C2. A familiar example is the case where $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ forms a dependent white-noise sequence as a result of dynamic conditional heteroskedasticity or some other type of nonlinear dependence. The finite-sample properties of tests for fractional processes that are linear in the weak sense will be considered in a later section.

As indicated before, when $\mathbf{X}$ satisfies (4) with $d \in (0, 1/2)$, the scaling factors that ensure convergence of the distributions of $\hat{\beta}_{1,n}$ and $\hat{\beta}_{2,n}$ to non-degenerate limits, as well as the limits themselves, depend on the value of d. When ε_0 is Gaussian, $\sqrt{n}v_1^{-1}\hat{\beta}_{1,n}$ and $\sqrt{n}v_2^{-1}(\hat{\beta}_{2,n}-3)$ are asymptotically standard normal if $d < 1/3$ and $d < 3/8$, respectively, where

$$v_r^2 := \sum_{k\in\mathbb{Z}} \mathrm{Cov}[H_{r+2}(\tilde{X}_k), H_{r+2}(\tilde{X}_0)],$$

$r \in \{1, 2\}$, and $H_q(\cdot)$, $q \geq 1$, is the Hermite polynomial of degree q with leading coefficient 1. As a result, a composite statistic based on normalized versions of $n\hat{\beta}_{1,n}^2$ and $n(\hat{\beta}_{2,n} - 3)^2$ is asymptotically chi-squared as long as $d < 1/3$. Asymptotic distributions are non-standard for values of d that do not belong to the indicated subintervals, with convergence rates that are functions of n and d. Specifically, $n^{3(1/2-d)}(c_d^{-1}\gamma_0)^{3/2}\hat{\beta}_{1,n}$ converges in distribution to $Z_3 - 3Z_1Z_2 + 2Z_1^3$ when $d > 1/3$, while $n^{2-4d}(c_d^{-1}\gamma_0)^2(\hat{\beta}_{2,n}-3)$ converges in distribution to $Z_4 - 4Z_1Z_3 + 12Z_1^2Z_2 - 3Z_2^2 - 6Z_1^4$ when $d > 3/8$ (Boutahar (2010)); here, Z_q, $q \geq 1$, stands for the value at time point 1 of a standard Hermite process of order q with self-similarity parameter $1 - (1/2 - d)q$, which is non-Gaussian for $q \geq 2$ (see Pipiras and Taqqu (2017, Ch. 4)). It is worth adding that, even when $\hat{\beta}_{1,n}$ and $\hat{\beta}_{2,n}$ are asymptotically normal at rate $\sqrt{n}$, precise estimation of their asymptotic variances is not a straightforward task, since the variances are values of infinite sums of autocovariances of polynomial functions of $\tilde{X}_t$.

To overcome the difficulties highlighted in the preceding two paragraphs, we suggest using an ARSB procedure to estimate the variances of $\hat{\beta}_{1,n}$ and $\hat{\beta}_{2,n}$ and to obtain P-values and/or critical values for tests based on S_n, K_n and G_n. As discussed in the next section, a key advantage of the ARSB in our setup is that it can be used for conducting inference on β_1 and β_2 without

knowledge or estimation of d.

3. Bootstrap Approximations

3.1. Bootstrap Procedure

The sampling distributions of the test statistic of interest under the relevant null hypothesis are approximated by means of the ARSB. The procedure is motivated by the observation that the fractional process defined by (4) admits the autoregressive representation

$$X_t - \mu = \sum_{j=1}^{\infty} \phi_j (X_t - \mu) + \varepsilon_t, \qquad t \in \mathbb{Z}, \tag{6}$$

where $\{\phi_j\}_{j=1}^{\infty}$ are real coefficients such that $(1 - \sum_{j=1}^{\infty} \phi_j z^j)(1 + \sum_{j=1}^{\infty} \theta_j z^j) = (1 - z)^d$ for all complex z with $|z| < 1$. Hence, $\phi_j \sim [\theta(1)\Gamma(-d)]^{-1} j^{-d-1}$ as $j \to \infty$ (for $d \neq 0$) and $\sum_{j=1}^{\infty} \phi_j^2 < \infty$. The core idea of the ARSB is to approximate (6) by a finite-order autoregressive model, which is then used as the basis for a semiparametric bootstrap scheme. If the order of the autoregressive approximation is allowed to increase simultaneously with the sample size n at an appropriate rate, the distribution of the process defined by (6) will be matched asymptotically.

To describe the ARSB more precisely, let $(\hat{\phi}_{1,n}, \ldots, \hat{\phi}_{p,n})$ and $\hat{\sigma}_n$ be the Yule–Walker estimators of the coefficients and noise standard deviation, respectively, of an autoregressive model of order p for $X_t - \mu$, where $p := p_n < n$ is a positive integer chosen as an increasing function of n. The ARSB version $\mathbf{X}^* := \{X_t^*\}_{t \in \mathbb{Z}}$ of $\mathbf{X}$ is defined via the equations

$$X_t^* - \bar{X}_n = \sum_{j=1}^{p} \hat{\phi}_{j,n} (X_{t-j}^* - \bar{X}_n) + \hat{\sigma}_n \varepsilon_t^*, \qquad t \in \mathbb{Z}, \tag{7}$$

where the random variables $\{\varepsilon_t^*\}_{t \in \mathbb{Z}}$ are i.i.d., conditionally on $\mathbf{X}^n$, with a suitably chosen common distribution Π_n (to be specified) having zero mean and unit variance. Note that $1 - \sum_{j=1}^{p} \hat{\phi}_{j,n} z^j \neq 0$ for all complex z with $|z| \leq 1$ and, consequently, the bootstrap process $\mathbf{X}^*$ is causal with respect to $\{\varepsilon_t^*\}_{t \in \mathbb{Z}}$ for all n.[2]

If $T_n := T_n(\mathbf{X}^n)$ is the statistic of interest (e.g., $\hat{\beta}_{1,n}$, $\hat{\beta}_{2,n}$, S_n, K_n, G_n), its bootstrap counterpart T_n^* is defined by the plug-in principle as $T_n^* := T_n(\mathbf{X}^{*n})$, where $\mathbf{X}^{*n} := \{X_t^*\}_{t=1}^n$. The bootstrap estimator of the sampling distribution of T_n is the conditional distribution of T_n^* given $\mathbf{X}_n$. Al-

[2] If $\mathbf{X}^*$ is defined using least-squares estimators, then it is causal with high probability for n large enough.

though this distribution is generally intractable, it can be approximated, to any desired degree of accuracy, by Monte Carlo simulation. Specifically, letting $\{T^*_{n,b}\}_{b=1}^B$ be B copies of T^*_n, which are independent conditionally on $\mathbf{X}^n$, the empirical distribution of the replicates $\{T^*_{n,b}\}_{b=1}^B$ provides an approximation to the bootstrap distribution of T_n.

3.2. Bootstrap Implementation

For ARSB-assisted tests to have good power properties, it is important to ensure that bootstrap data are generated in a way which reflects the null hypothesis under test (symmetry, mesokurtosis, normality) regardless of whether the null hypothesis is true. To this end, when testing $\beta_1 = 0$ using the skewness statistic (1), we take the distribution Π_n of ε^*_0, conditional on $\mathbf{X}^n$, to be the symmetrized empirical distribution of the standardized residuals $\{\tilde{\varepsilon}_{t,n} := \hat{s}_n^{-1}(\hat{\varepsilon}_{t,n} - \hat{\varepsilon}^\circ_n)\}_{t=p+1}^n$, where

$$\hat{\varepsilon}_{t,n} := X_t - \bar{X}_n - \sum_{j=1}^{p} \hat{\phi}_{j,n}(X_{t-j} - \bar{X}_n), \qquad t = p+1, p+2, \ldots, n,$$

$\hat{\varepsilon}^\circ_n := (n-p)^{-1}\sum_{t=p+1}^n \hat{\varepsilon}_{t,n}$ and $\hat{s}_n^2 := (n-p)^{-1}\sum_{t=p+1}^n (\hat{\varepsilon}_{t,n} - \hat{\varepsilon}^\circ_n)^2$; that is to say, Π_n is the distribution that places mass $[2(n-p)]^{-1}$ at each of $\pm\tilde{\varepsilon}_{p+1,n}, \ldots, \pm\tilde{\varepsilon}_{n,n}$. For tests of $\beta_2 = 0$ or $(\beta_1, \beta_2) = (0, 3)$ based on the kurtosis statistic (2) or the composite statistic (3), we take Π_n to be the standard normal distribution. For any given Π_n, bootstrap replicates $\mathbf{X}^{*n}$ are obtained by setting $X^*_1 = \cdots = X^*_p = \bar{X}_n$, generating replicates $\{X^*_t\}_{t=p+1}^{n+\bar{n}}$ recursively according to (7), for some large positive integer $\bar{n}$, and then discarding the first $\bar{n} - p$ replicates to eliminate any start-up effect.

Another important consideration is the choice of the order p of the autoregressive approximation. Following common practice, a data-dependent order is selected from the set of positive integers that are no greater than a pre-assigned maximal order $M := M_n < n$ so as to minimize Akaike's information criterion

$$\mathrm{AIC}(p) := \log\left((n-M)^{-1}\sum_{t=M+1}^{n} \hat{\varepsilon}^2_{t,n}\right) + 2pn^{-1}. \tag{8}$$

The AIC-based choice of p is known to be asymptotically optimal for any $|d| < 1/2$, in the sense defined by Shibata (1980), provided $M \to \infty$ and $(n^{-1}\log n)^{1-2\bar{d}}\xi_M^{-1}M \to 0$ as $n \to \infty$, where $\bar{d} := \max\{0, d\}$ and ξ_M is the smallest eigenvalue of the $M \times M$ matrix with (i,j) entry γ_{i-j} (Poskitt (2007, Theorem 9)).

The studentization of the original and bootstrap versions of $\hat{\beta}_{1,n}$ and $\hat{\beta}_{2,n}$ also requires atten-

tion. As pointed out in Section 2, this is not a trivial task, since the appropriate normalizing factors for $\hat{\beta}_{1,n}$ and $\hat{\beta}_{2,n}$ are not the same for different values of d. Furthermore, these factors often involve infinite series generated by the autocovariance sequences of polynomial functions of $\tilde{X}_t$, estimation of which by means of familiar truncated empirical versions that rely on tapering and/or thresholding is a challenging problem. To deal with these difficulties, we propose studentizing $\hat{\beta}_{1,n}$ and $\hat{\beta}_{2,n}$ using variance estimates obtained by the ARSB. Since the sampling distributions of the resulting statistics are also estimated by the ARSB, this approach necessitates the use of a nested double-bootstrap procedure.

Focusing on S_n, the procedure for obtaining an approximate P-value for a test that rejects $\beta_1 = 0$ for large values of $|S_n|$ comprises the following steps (the procedures for tests based on K_n and G_n are analogous):

(i) For a sufficiently large positive integer B, let $\mathbf{X}_1^{*n}, \ldots, \mathbf{X}_B^{*n}$ be copies of the bootstrap sample $\mathbf{X}^{*n}$ that are conditionally independent, given $\mathbf{X}^n$. If $\hat{\beta}^*_{1,n,b}$, $b \in \{1, 2, \ldots, B\}$, is the bootstrap counterpart of $\hat{\beta}_{1,n}$ based on $\mathbf{X}_b^{*n}$, then an approximation to the bootstrap estimator of the variance of $\hat{\beta}_{1,n}$ is given by

$$\hat{\omega}^{*2}_{1,n} := B^{-1} \sum_{b=1}^{B} \hat{\beta}^{*2}_{1,n,b} - \left(B^{-1} \sum_{b=1}^{B} \hat{\beta}^*_{1,n,b} \right)^2 .$$

Thus, an operational version of S_n is obtained as $S_n = \hat{\beta}_{1,n}/\hat{\omega}^*_{1,n}$.

(ii) For each $b \in \{1, 2, \ldots, B\}$, let $\mathbf{X}_b^{**n}$ be the bootstrap version of $\mathbf{X}_b^{*n}$ obtained by applying the ARSB to $\mathbf{X}_b^{*n}$ in place of $\mathbf{X}^n$ (using the same autoregressive order p that is used for $\mathbf{X}^{*n}$). Further, for a sufficiently large positive integer N, let $\mathbf{X}_{b,1}^{**n}, \ldots, \mathbf{X}_{b,N}^{**n}$ be copies of $\mathbf{X}_b^{**n}$ that are conditionally independent, given $\mathbf{X}^{*n}$ and $\mathbf{X}^n$. An approximation to the bootstrap variance of $\hat{\beta}^*_{1,n,b}$ is given by

$$\hat{\omega}^{**2}_{1,n} := N^{-1} \sum_{i=1}^{N} \hat{\beta}^{**2}_{1,n,i} - \left(N^{-1} \sum_{i=1}^{N} \hat{\beta}^{**}_{1,n,i} \right)^2 ,$$

where $\hat{\beta}^{**}_{1,n,i}$ is the bootstrap counterpart of $\hat{\beta}_{1,n}$ based on $\mathbf{X}_{b,i}^{**n}$. Hence, the bootstrap counterpart of S_n based on $\mathbf{X}_b^{*n}$ is obtained as $S^*_{n,b} := \hat{\beta}^*_{1,n,b}/\hat{\omega}^{**}_{1,n}$.

(iii) An approximate P-value for a (symmetrical) test that rejects for large absolute values of $S_n = \hat{\beta}_{1,n}/\hat{\omega}^*_{1,n}$ is given by $\hat{P}^*_{1,n} := B^{-1} \sum_{b=1}^{B} \mathbb{I}(|S^*_{n,b}| > |S_n|)$, where $\mathbb{I}(\cdot)$ is the indicator

function. The hypothesis $\beta_1 = 0$ is rejected at nominal level $\alpha \in (0,1)$ if $\hat{P}^*_{1,n} \leq \alpha$.

An alternative to the approach outlined above, which avoids the application of the nested bootstrap, is to rely on ARSB-assisted tests based on non-studentized statistics. This amounts to rejecting the hypotheses $\beta_1 = 0$ and $\beta_2 = 3$ at nominal level α if $\tilde{P}^*_{1,n} := B^{-1}\sum_{b=1}^{B}\mathbb{I}(|\hat{\beta}^*_{1,n,b}| > |\hat{\beta}_{1,n}|) \leq \alpha$ and $\tilde{P}^*_{2,n} := B^{-1}\sum_{b=1}^{B}\mathbb{I}(|\hat{\beta}^*_{2,n,b} - 3| > |\hat{\beta}_{2,n} - 3|) \leq \alpha$, respectively, where $\{\hat{\beta}^*_{r,n,b}\}_{b=1}^{B}$, $r \in \{1,2\}$, are bootstrap analogues of $\hat{\beta}_{r,n}$ computed from $\mathbf{X}_1^{*n}, \ldots, \mathbf{X}_B^{*n}$. Moreover, rather than combining the unscaled $\hat{\beta}^2_{1,n}$ and $\hat{\beta}^2_{2,n}$ into a statistic analogous to G_n, one may instead construct an induced test that rejects $(\beta_1, \beta_2) = (0,3)$ when either $\tilde{P}^*_{1,n}$ or $\tilde{P}^*_{2,n}$ are sufficiently small. To control the probability of erroneously rejecting symmetry or mesokurtosis when both conditions hold, it is advisable that the individual P-values be appropriately adjusted. In what follows, a Šidák-type adjustment is employed, and $(\beta_1, \beta_2) = (0,3)$ is rejected at level α if $\min\{1-(1-\tilde{P}^*_{1,n})^2, 1-(1-\tilde{P}^*_{2,n})^2\} \leq \alpha$.[3] For convenience, and with some abuse of terminology, we hereafter take the quantity on the left-hand side of the previous inequality to represent the ARSB P-value for the induced normality test.

Finally, it is worth pointing out that scaling of $\hat{\beta}_{1,n}$ and $\hat{\beta}_{2,n}$ by deterministic constants that are functions of n and d, although necessary for obtaining non-degenerate asymptotic distributions, is not required when using the bootstrap procedure described above. Bootstrap P-values are invariant to scaling by such constants when these are applied simultaneously to the original statistics and their bootstrap counterparts. This is especially useful in our case, since the choice of appropriate scaling constants requires knowledge about the value of the memory parameter. Coupled with the use of bootstrap estimates of the variances of $\hat{\beta}_{1,n}$ and $\hat{\beta}_{2,n}$ in the construction of studentized test statistics, this ensures that ARSB-assisted tests can be implemented in the same way irrespective of the data-generating process exhibiting short memory, antipersistence or long memory.

3.3. Bootstrap Validity

To ensure that ARSB approximations are asymptotically valid, the following condition on the autoregressive order p used in the definition of the bootstrap process $\mathbf{X}^*$ is imposed:

C3: $p \to \infty$ and $p = \mathcal{O}\left(\{\log n\}^{\ell}\right)$ as $n \to \infty$, for some $\ell \geq 1$.

[3] A similar approach can be adopted when using the studentized statistics $|\hat{\beta}_{1,n}|/\hat{\omega}^*_{1,n}$ and $|\hat{\beta}_{2,n} - 3|/\hat{\omega}^*_{2,n}$, instead of relying on G_n and its ARSB counterpart.

The requirement C3, although perhaps stronger than necessary, is appealing because the specified rate of increase of p is independent of d. It is satisfied, with probability 1, by a data-dependent order p obtained as the minimizer of (8), as long as M grows with n at rate $\mathcal{O}\big(\{\log n\}^{\ell}\big)$ (Psaradakis (2016, Proposition 2)). For instance, one may set $M = \lceil C_1(\log n)^m \rceil$, for some $C_1 > 0$ and $m \geq 1$, where $\lceil \cdot \rceil$ denotes the ceiling function.

The following proposition shows that the ARSB is asymptotically valid for approximating the sampling distributions of $\hat{\beta}_{1,n}$ and $\hat{\beta}_{2,n}$ for data from a Gaussian, fractional process whose memory parameter lies in $(-1/2, 1/2)$. Here, $F_{r,n}$, $r \in \{1,2\}$, stands for the distribution function of $\hat{\beta}_{r,n}$ (which is assumed to be appropriately centered and scaled to have a non-degenerate asymptotic distribution) and $F^*_{r,n}$ stands for the conditional distribution function, given $\mathbf{X}^n$, of the bootstrap counterpart $\hat{\beta}^*_{r,n}$ of $\hat{\beta}_{r,n}$ based on $\mathbf{X}^{*n}$ (similarly centered and scaled). Closeness between the two distribution functions is quantified in terms of their Mallows–Wasserstein distance, defined as $\varrho(F^*_{r,n}, F_{r,n}) := \{\int_0^1 [F^{*-1}_{r,n}(\tau) - F^{-1}_{r,n}(\tau)]^2 \mathrm{d}\tau\}^{1/2}$, where, for any distribution function F on the real line, $F^{-1}(\tau) := \inf\{x : F(x) \geq \tau\}$ is the associated quantile function. As is well known, ϱ metrizes pointwise convergence on a dense subset of the real line. Note that $F_{r,n}$ and $F^*_{r,n}$ (as well as their weak limits) depend on d, although this is not explicitly indicated in the notation.

Suppose that $\mathbf{X}$ satisfies (4), C1–C3 hold and ε_0 is normally distributed. Then, for $r \in \{1,2\}$, $\varrho(F^*_{r,n}, F_{r,n}) \to 0$ almost surely as $n \to \infty$.

Some comments are in order.

1. The statistics $\hat{\beta}_{r,n}$, $r \in \{1,2\}$, satisfy the smoothness conditions of Lemma 1 of Poskitt (2008) and, hence, his Assumption 4. Thus, in light of his Remark 2 regarding the use of mean-corrected replicates $X^*_t - \bar{X}_n$ in (7), it follows from his Theorem 2 that $\varrho(F^*_{r,n}, F_{r,n}) \to 0$ almost surely as $n \to \infty$ under $\Pi_n = \tilde{Q}_n$, where $\tilde{Q}_n$ is the empirical distribution of $\{\tilde{\varepsilon}_{t,n}\}_{t=p+1}^n$. The use of $\Pi_n = \mathrm{N}$, N being the standard normal distribution, when Gaussianity of ε_0 is assumed (see the discussion at the beginning of Section 3.2) is asymptotically justified by the fact that N and $\tilde{Q}_n$ are close in ϱ-distance. This can be seen by noting that, by the definition of ϱ, $\{\varrho(\tilde{Q}_n, Q_n)\}^2 \leq (n-p)^{-1} \sum_{t=p+1}^n (\tilde{\varepsilon}_{t,n} - \sigma^{-1}\varepsilon_t)^2 =: U_n$, where Q_n is the empirical distribution of $\{\varepsilon_t\}_{t=p+1}^n$. By calculations similar to those in the proof of Theorem 2 of Poskitt (2008), it can be deduced that $U_n \to 0$ almost surely. Since $\varrho(\tilde{Q}_n, \mathrm{N}) \leq \varrho(\tilde{Q}_n, Q_n) + \varrho(Q_n, \mathrm{N})$, this and Lemma 8.4

of Bickel and Freedman (1981) ensure that $\varrho(\tilde{Q}_n, \mathrm{N}) \to 0$ almost surely as $n \to \infty$.

2. Since convergence in ϱ-distance also implies convergence of second-order moments (cf. Bickel and Freedman (1981, Lemma 8.3)), it follows from Proposition 3.3 that the variance of the conditional distribution of $\hat{\beta}^*_{r,n}$, $r \in \{1, 2\}$, given $\mathbf{X}^n$, estimates consistently the variance of $\hat{\beta}_{r,n}$. While analytical determination of this bootstrap variance estimator is intractable, its Monte Carlo approximation $\hat{\omega}^{*2}_{r,n}$ discussed in Section 3.2 has an approximation error that vanishes almost surely as $B \to \infty$ (cf. Bickel and Freedman (1981, Lemma 8.4)).

3. The use of the variance estimators $\hat{\omega}^{**2}_{r,n}$, $r \in \{1, 2\}$, at the second level of the nested double-bootstrap procedure described in Section 3.2 is justified asymptotically by reasoning similar to that used in the previous comment. Indeed, when considering the validity of ARSB estimators of the variance of $\hat{\beta}^*_{r,n}$, the usual difficulty of controlling the error induced by approximating an infinite-order autoregressive process with a finite-order one does not arise. This is because $\mathbf{X}^{*n}$ and its bootstrap counterpart $\mathbf{X}^{**n}$ are both causal autoregressive processes of the same order p. As in the case of $\hat{\omega}^{*2}_{r,n}$, the Monte Carlo approximation $\hat{\omega}^{**2}_{r,n}$ to the bootstrap variance of $\hat{\beta}^*_{r,n}$ is subject only to a simulation error that becomes negligible as $N \to \infty$. These results, together with Proposition 3.3 and the conditional version of Slutsky's theorem, justify the use of ARSB tests based on studentized skewness and kurtosis statistics for large enough n, B and N.

4. Under a resampling scheme in which Π_n is the symmetrized empirical distribution of $\{\tilde{\varepsilon}_{t,n}\}_{t=p+1}^n$, adopted when testing the hypothesis $\beta_1 = 0$ on its own, asymptotic validity of the ARSB for $\hat{\beta}_{1,n}$ follows by the same argument as in Proposition 1 of Psaradakis (2016). In this case, it suffices to assume that C1–C3 hold, the distribution of ε_0 is symmetric and the weak limit of $F_{1,n}$ is non-degenerate. In view of the results in Ho (2002), the latter requirement is met for $d \neq 0$ when C2 holds with $\nu = 2$, since (with appropriate scaling) $\hat{\beta}_{1,n}$ has the same asymptotic distribution as $n^{-1} \sum_{t=1}^n [\gamma_0^{-1/2}(X_t - \bar{X}_n)]^3$ under $\beta_1 = 0$ and the function $(x, y) \mapsto \gamma_0^{-3/2}(x - y)^3$ satisfies the regularity conditions of Assumption A2 in the aforementioned paper.

5. The ARSB approach may be used to test hypotheses about any postulated values of β_1 and β_2, not only those associated with Gaussian distributions. However, choosing the distribution Π_n of the bootstrap noise so that the bootstrap data $\mathbf{X}^{*n}$ have the postulated skewness and kurtosis is not always an easy task. A viable alternative is to rely on bootstrap confidence intervals (or

tests) for β_1 and β_2 that are constructed under a resampling scheme in which $\Pi_n = \tilde{Q}_n$. In the case of β_1, for instance, using the double-bootstrap algorithm described in Section 3.2 (with $\Pi_n = \tilde{Q}_n$) to estimate the variances of $\hat{\beta}_{1,n}$ and $\hat{\beta}^*_{1,n}$, a $100(1-\alpha)\%$ symmetric confidence interval for β_1 takes the form $[\hat{\beta}_{1,n} - \kappa^*_{1,\alpha}\hat{\omega}^*_{1,n}, \hat{\beta}_{1,n} + \kappa^*_{1,\alpha}\hat{\omega}^*_{1,n}]$, where $\kappa^*_{1,\alpha}$ is the $(1-\alpha)$th quantile of the empirical distribution of $\{|\hat{\beta}^*_{1,n,b} - \hat{\beta}_{1,n}|/\hat{\omega}^{**}_{1,n}\}_{b=1}^B$. A simpler, though potentially less accurate, confidence interval is obtained as $[\hat{\beta}_{1,n} - \bar{\kappa}^*_{1,\alpha}, \hat{\beta}_{1,n} + \bar{\kappa}^*_{1,\alpha}]$, where $\bar{\kappa}^*_{1,\alpha}$ is the $(1-\alpha)$th quantile of the empirical distribution of $\{|\hat{\beta}^*_{1,n,b} - \hat{\beta}_{1,n}|\}_{b=1}^B$. In either case, for some specified value β_1^0, the null hypothesis $\beta_1 = \beta_1^0$ is rejected at level α if β_1^0 is not contained in the confidence interval. This is equivalent to constructing an α-level, two-sided test of $\beta_1 = \beta_1^0$ using a bootstrap test statistic which is defined in terms of the difference $\hat{\beta}^*_{1,n} - \hat{\beta}_{1,n}$ instead of $\hat{\beta}^*_{1,n} - \beta_1^0$, where $\hat{\beta}^*_{1,n}$ is based on bootstrap data that are generated without imposing the null hypothesis.

4. Monte Carlo Experiments

In this section, we report and discuss the results of an extensive simulation study of the finite-sample properties of bootstrap tests based on the empirical coefficients of skewness and kurtosis under various data-generating processes.

4.1. Experimental Design

The Monte Carlo experiments are based on artificial data from the FARIMA process defined by the equations

$$X_t = (1-L)^{-d}(1-\varphi L)^{-1}(1+\vartheta L)\varepsilon_t, \qquad t \in \mathbb{Z},$$

with $d \in \{-0.4, -0.2, 0, 0.2, 0.4\}$ and the following (φ, ϑ):

M1: $(\varphi, \vartheta) = (0.5, 0)$;

M2: $(\varphi, \vartheta) = (0, 0.5)$;

M3: $(\varphi, \vartheta) = (0.6, 0.3)$.

The common distribution of the i.i.d. random variables $\{\varepsilon_t\}$ is either standard normal (labelled N in the tables that follow) or generalized lambda with distribution function F_ε satisfying $F_\varepsilon^{-1}(\tau) = \lambda_1 + \lambda_2^{-1}\{\tau^{\lambda_3} - (1-\tau)^{\lambda_4}\}$, $0 < \tau < 1$ (recentered and rescaled to have zero mean and unit variance). Most of the values of $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ used in the experiments are taken from Bai and Ng (2005) and can be found in Table 1, alongside the associated coefficients of skew-

ness and kurtosis; distributions S1–S3 are leptokurtic but symmetric, whereas A1–A4 are both leptokurtic and asymmetric.

Table 1: Noise Distributions

	λ_1	λ_2	λ_3	λ_4	Skewness	Kurtosis
N	–	–	–	–	0.0	3.0
S1	0.00000	-1.00000	-0.08000	-0.08000	0.0	6.0
S2	0.00000	-0.39791	-0.16000	-0.16000	0.0	11.6
S3	0.00000	-1.00000	-0.22000	-0.22000	0.0	39.1
A1	0.00000	-1.00000	-0.00750	-0.03000	1.5	7.5
A2	0.00000	-1.00000	-0.10090	-0.18020	2.0	21.1
A3	0.00000	-1.00000	-0.00100	-0.13000	3.2	23.8
A4	0.00000	-1.00000	-0.00010	-0.17000	3.8	40.7

For a given design point, 1,000 independent realizations of $\{X_t\}$ of length $100+n$, $n \in \{100, 200, 400\}$, are generated, discarding the first 100 data points in each case to eliminate start-up effects. For each of these realizations, we consider testing three null hypotheses, namely $\beta_1 = 0$ (symmetry), $\beta_2 = 3$ (mesokurtosis) and $\beta_1 = \beta_2 - 3 = 0$ (normality), using the statistics S_n, K_n and G_n, respectively. The P-values for the tests are obtained using the nested double-bootstrap procedure described in Section 3.2, with $N = 100$ inner-bootstrap replications (to compute the ARSB variances of the bootstrap counterparts of $\hat{\beta}_{1,n}$ and $\hat{\beta}_{2,n}$) and $B = 399$ outer-bootstrap replications (to compute ARSB estimates of the variances of $\hat{\beta}_{1,n}$ and $\hat{\beta}_{2,n}$ and to estimate the null sampling distributions of the studentized statistics).[4] In addition, we also consider tests based on the non-studentized statistics $\hat{\beta}_{1,n}$ and $\hat{\beta}_{2,n}$, and the associated induced test for normality, implemented in the way described in Section 3.2 with $B = 399$. The order of the autoregressive approximation is chosen by AIC(p), with maximal allowable order $M = \lceil (\log n)^2/2 \rceil$. The resulting model is fitted by least-squares; this is preferred over the Yule–Walker method because, although the difference between the two coefficient estimators is asymptotically negligible (cf. Poskitt (2007, Corollary 1)), least-squares estimators typically have smaller finite-sample bias.

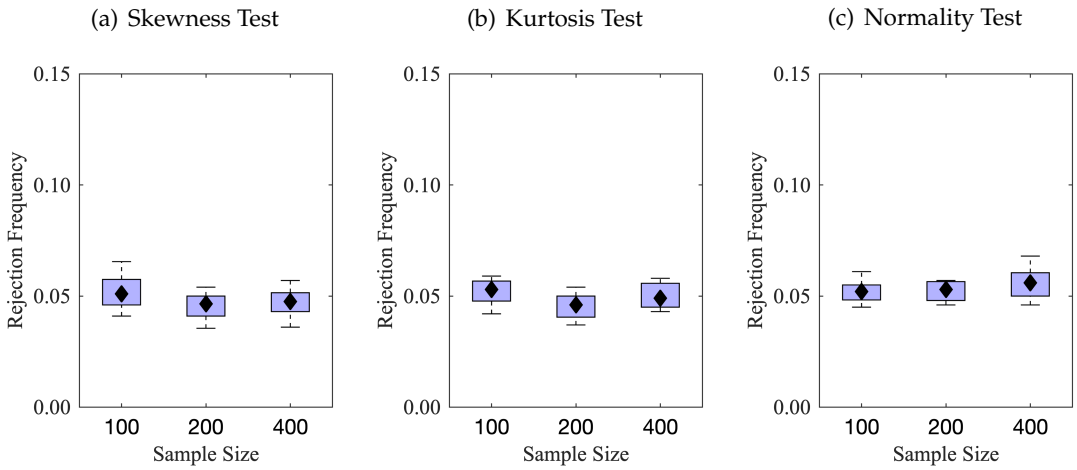
4.2. Simulation Results

Results over all design points that satisfy the null hypothesis of either symmetry (60 points), mesokurtosis (15 points) or normality (15 points) are summarized graphically in Figure 1. This shows box plots of the Monte Carlo rejection frequencies of ARSB-assisted tests of nominal

[4] We note that using a larger number of bootstrap replications does not change the results substantially. This is consistent with the results of Hall (1986) and Jöckel (1986) that simulation-based inference procedures can be accurate for relatively small values of the simulation size.

level $\alpha = 0.05$ based on S_n, K_n and G_n. The top and bottom of each colored box represent the 25th and 75th percentiles, respectively, of the rejection frequencies, the black diamond inside the box indicates the median value, and the whiskers indicate the 10th and 90th percentiles. Detailed results for individual design points and each of the three tests can be found in Tables 3–11 in the Supplementary Material.

Figure 1: Observed Level of Tests (Studentized Statistics)

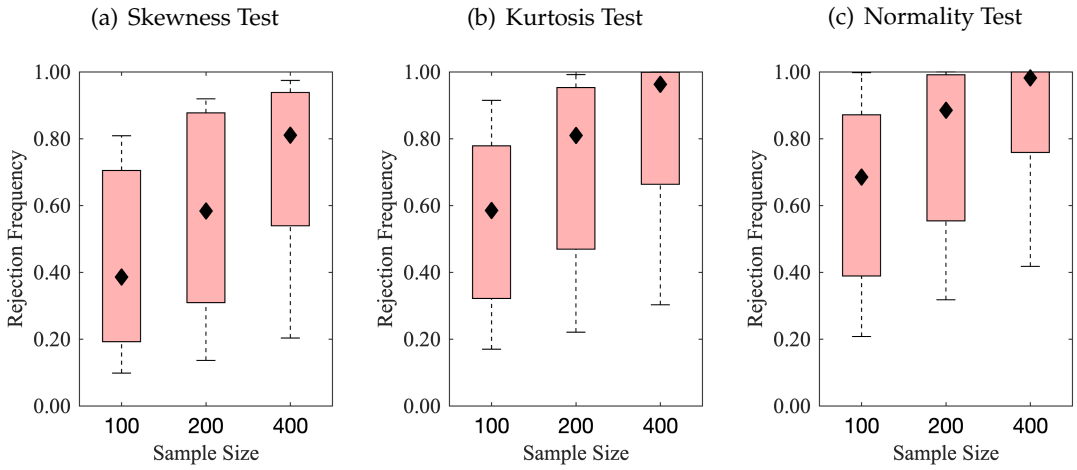


The differences between the experimentally observed and nominal probabilities of Type I error are small for the ARSB-assisted tests. Inspection of the results for individual design points reveals that level accuracy is maintained across all three combinations of the parameters (φ, ϑ) and all five values of the memory parameter d. Even in the few cases where some level distortion is observed, its magnitude is not large enough to render the tests unattractive for application. It is worth noting that, unless the underlying distribution has Gaussian-like tails, tests based on $\hat{\beta}_{2,n}$ are often reported to exhibit substantial level distortion under i.i.d. or short-memory conditions when asymptotic critical values are used (see, e.g., Bai and Ng (2005) and Peng-Zhou and Song (2024)). Employing the ARSB to estimate the variance of $\hat{\beta}_{2,n}$ and approximate the null distribution of the resulting studentized statistic K_n, we obtain a test for mesokurtosis which maintains accurate level performance for data from long-memory, short-memory or antipersistent processes.

Box plots of the Monte Carlo rejection frequencies of tests of nominal level $\alpha = 0.05$ over all design points that do not satisfy the null hypothesis under test are shown in Figure 2. Detailed results for individual design points and each of the three tests can be found in Tables 3–11 in the Supplementary Material. It is clear that power improves very quickly as the sample size

n increases. Deviations from symmetry are the least effectively detected, particularly when long memory is present. The test for normality tends to reject the null hypothesis with high probability, although it is less successful at doing so when the noise distribution is symmetric and/or when the autocovariances of the data decay slowly ($d = 0.4$). The presence of strong dependence ($d = 0.4$) also has an adverse effect on the power of the test designed to detect deviations from mesokurtosis.

Figure 2: Observed Power of Tests (Studentized Statistics)



These findings regarding test power are not surprising once the symmetry and tail-heaviness/peakedness characteristics of the one-dimensional marginal distribution of $\{X_t\}$ implied by the experimental design are considered. The coefficients of skewness and kurtosis of this distribution for selected values of d are reported in Tables 1 and 2 in the Supplementary Material—these are approximated numerically by averaging the values of $\hat{\beta}_{1,n}$ and $\hat{\beta}_{2,n}$ over 100 simulated time series, each of length $n = 10^6$. For any combination of (φ, ϑ) and non-Gaussian noise, the distribution of X_0 exhibits less asymmetry and considerably less leptokurtosis when $d = 0.4$ compared to when $d = -0.4$ or $d = 0$.

The computationally less intensive approach based on non-studentized statistics is not, as might be expected, significantly disadvantaged. This can be seen in Figures 3 and 4, which contain box plots analogous to those described previously for 0.05-level ARSB-assisted tests. The method delivers tests which generally have larger level distortions than their counterparts based on S_n, K_n and G_n, particularly when testing the hypothesis $\beta_1 = 0$. The differences between the observed rejection rates and the nominal level, however, are not of a magnitude to suggest that the use of the tests is inadvisable, especially when $n \geq 200$. Moreover, de-

partures from symmetry, mesokurtosis and normality are detected successfully at rates often comparable to those of their double-bootstrap counterparts and, for some design points, somewhat higher (with the caveat that power comparisons are not straightforward because the two groups of tests do not share the same finite-sample levels).

Figure 3: Observed Level of Tests (Non-studentized Statistics)

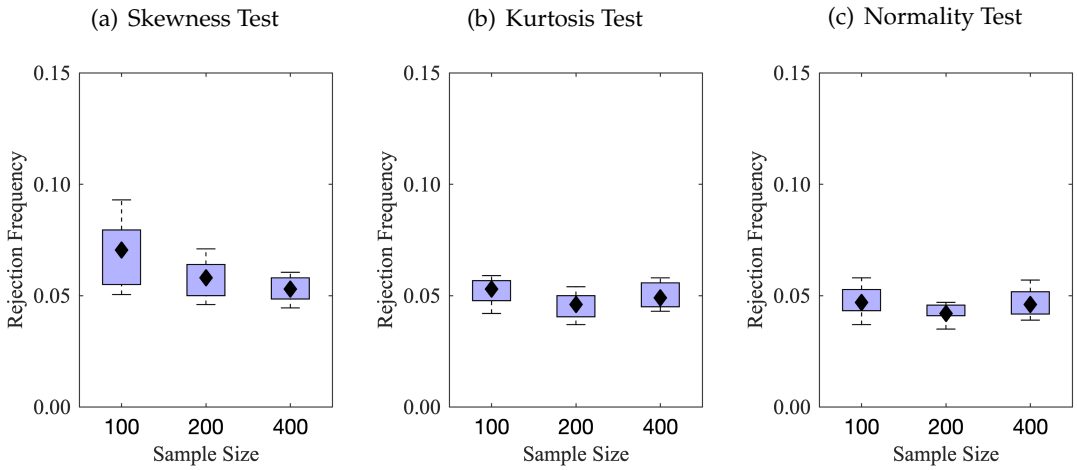
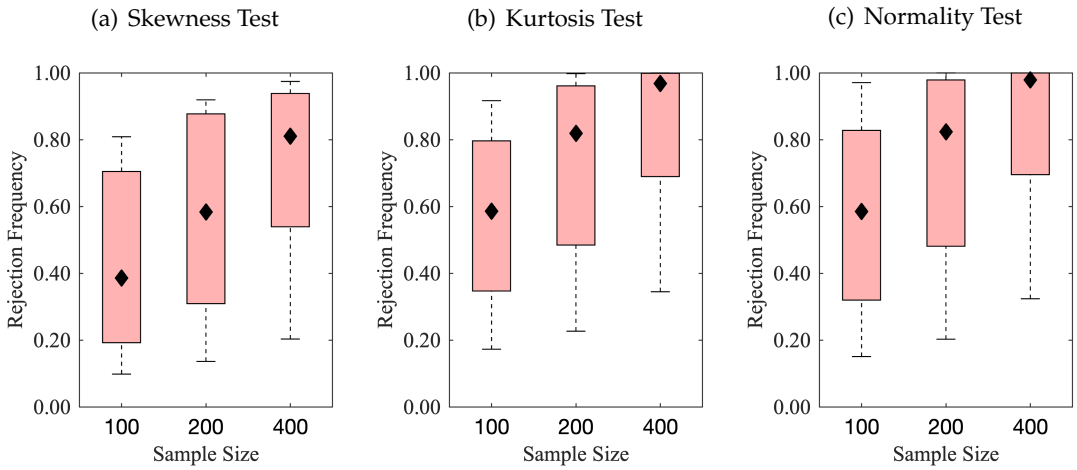


Figure 4: Observed Power of Tests (Non-studentized Statistics)



The rejection frequencies of tests for individual design points, reported in Tables 21–29 in the Supplementary Material, exhibit the same general patterns described in the previous paragraphs for tests based on studentized statistics. They indicate that ARSB-assisted inference based on non-studentized skewness and kurtosis statistics offers a viable option for avoiding the computational cost of nested bootstrapping, although typically at the expense of reduced level accuracy.

4.3. Nonlinearity

We conclude the simulation study by discussing the results from additional experiments based on data-generating processes which do not satisfy the assumption of linearity (in the strict sense). Specifically, we consider artificial data generated according to the fractional process $X_t = (1-L)^{-d}Y_t$, $t \in \mathbb{Z}$, where Y_t is a short-memory process satisfying one of the following (with ε_t as in the earlier experiments):

M4: $Y_t = (0.9Y_{t-1} + \varepsilon_t)\mathbb{I}(|Y_{t-1}| \leq 1) - (0.3Y_{t-1} - 2\varepsilon_t)\mathbb{I}(|Y_{t-1}| > 1)$;

M5: $Y_t = 0.3Y_{t-2}\varepsilon_{t-1} + \varepsilon_t$;

M6: $Y_t = 0.5Y_{t-1} + \eta_t\varepsilon_t, \quad \eta_t^2 = 0.1 + (0.05\varepsilon_{t-1}^2 + 0.85)\eta_{t-1}^2$.

The fractionally differenced process $\{(1-L)^d X_t\}$ has a self-exciting threshold autoregressive structure under M4, a bilinear structure under M5, and an autoregressive structure with autoregressive conditional heteroskedasticity under M6. Unlike the FARIMA processes considered in Sections 4.1 and 4.2, $\{X_t\}$ does not admit an autoregressive representation like (6) with respect to an i.i.d. noise sequence in these cases. However, as long as the spectral density of $\{Y_t\}$ does not vanish anywhere, a well-defined autoregressive representation for $\{X_t\}$ exists, albeit one with respect to uncorrelated rather than independent noise, namely the innovation process associated with the Wold decomposition of $\{X_t\}$ (cf. Bondon and Palma (2007, Theorem 3)). Another noteworthy feature of M4, M5 and M6 is that $\{Y_t\}$ is a non-Gaussian process even when ε_0 is normally distributed.[5] The coefficients of skewness and kurtosis of X_0 for selected values of d (approximated numerically by averaging the values of $\hat{\beta}_{1,n}$ and $\hat{\beta}_{2,n}$ over 100 simulated time series, each of length $n = 10^6$) can be found in Tables 1 and 2 in the Supplementary Material. The distribution of X_0 is leptokurtic in all cases, as well as asymmetric when ε_0 has a skewed distribution.

Tables 12–20 in the Supplementary Material contain the Monte Carlo rejection frequencies of bootstrap tests based on the statistics S_n, K_n and G_n. Nonlinearity in the form of threshold (M4) or bilinear (M5) dynamics does not appear to have an adverse effect on the properties of ARSB-assisted tests. Departures from mesokurtosis and normality are detected with high frequency, except in cases where the coefficient of kurtosis of X_0 is below 4. The probability

[5] The fourth moment of Y_0 is finite in all three cases, while the third moment is zero when ε_0 is symmetrically distributed.

of Type I error of the test for symmetry is controlled successfully in the majority of design points. There are a few design points for which the test is liberal, the highest observed rejection frequencies being 0.106 under M4 and 0.103 under M5 (the only exceptions being three design points under M5 with $d = 0.4$ and $n \geq 200$ where rejection frequencies reach a maximum of 0.138). While rejection frequencies under asymmetric noise distributions are not always high, this reflects the fact that the distribution of X_0 exhibits only mild asymmetry for some design points.

Comparison of the results in Tables 18–20 in the Supplementary Material with those in Tables 3–5 reveals that nonlinear dependence in the noise of the FARIMA process in the form of conditional heteroskedasticity (M6) generally leads to higher rejection frequencies for mesokurtosis and normality tests. This is not surprising, given that the coefficient of kurtosis of $\eta_0\varepsilon_0$ is at least as large as that of ε_0. The observed level of the symmetry test also increases slightly, but not to an extent that compromises its practical usefulness (the highest rejection frequency is 0.137, for $d = 0.4$, $n = 400$ and S2 noise).[6]

Qualitatively similar results are obtained using bootstrap tests based on non-studentized statistics, the rejection frequencies of which are reported in Tables 30–38 in the Supplementary Material. The finite-sample behavior of the tests generally mirrors that of their counterparts based on S_n, K_n and G_n. As in the case of FARIMA processes, studentization is advantageous, particularly when testing the hypothesis $\beta_1 = 0$, resulting in tests that exhibit smaller level distortions for the two smaller sample sizes.

In summary, our findings indicate that ARSB-assisted tests perform well for fractional linear processes whose innovations are uncorrelated but not independent. While asymptotic correctness of the tests is not guaranteed under such conditions, the non-Gaussian behavior typically associated with nonlinear dependence, when sufficiently pronounced, allows tests based on $\hat{\beta}_{1,n}$ and $\hat{\beta}_{2,n}$ to reliably detect asymmetry and leptokurtosis.

5. Empirical Application

Model-free measures of the volatility of financial returns based on high-frequency data have attracted considerable attention in empirical finance and financial econometrics. In this section, we examine the distributional properties of such measures, focusing on the symmetry, kurtosis

[6] We note that similar results were obtained with $\eta_t^2 = \exp(-0.05\varepsilon_{t-1} + 0.1|\varepsilon_{t-1}| + 0.85 \log \eta_t^2)$.

and normality of their marginal distribution. More specifically, we consider two nonparametric measures of daily volatility constructed from 5-minute returns: the standard realized volatility (RV) measure (square root of sum of squared returns) and the median realized volatility (MRV) measure, that is, the square root of the MedRV measure of Andersen et al. (2012). RV is extensively used and generally performs well in data-based comparisons of volatility measures (see Liu et al. (2015)). MRV mirrors RV in cumulating a sum of squared returns, but achieves robustness with respect to discontinuities, jumps, outliers and microstructure noise through local truncation of absolute returns. RV and related measures are widely reported to exhibit long memory and to have asymmetric and leptokurtic marginal distributions, which become approximately Gaussian after a logarithmic transformation (see, e.g., Andersen et al. (2001), Andersen et al. (2003)).

Our analysis focuses on the volatility of 10 Dow Jones stocks: Amgen (AMGN), Caterpillar (CAT), Goldman Sachs (GS), Honeywell International (HON), Johnson & Johnson (JNJ), 3M (MMM), Merck & Co. (MRK), Travelers Companies (TRV), UnitedHealth Group (UNH), and Walmart (WMT). The volatility time series, obtained from the VOLARE database (Cipollini et al. (2026)), comprise $n = 1064$ observations covering the period from January 4, 2021 to March 28, 2025.

Tables 2 and 3 contain P-values for tests of $\beta_1 = 0$, $\beta_2 = 3$ and $(\beta_1, \beta_2) = (0, 3)$ based on the statistics S_n, K_n and G_n, respectively, for RV, MRV and their logarithms. These are obtained using the nested double-bootstrap procedure described in Section 3.2, with outer-bootstrap replications $B = 10,000$ and inner-bootstrap replications $N = 500$. The order of the autoregressive sieve is selected by minimizing $\mathrm{AIC}(p)$ over $1 \le p \le \lceil (\log n)^2/2 \rceil$.

A semiparametric estimate $(\hat{d})$ of the memory parameter of each time series and the associated estimated asymptotic standard error $(SE(\hat{d}))$ are also reported in the tables. These are obtained using the local pseudo-Gaussian Whittle estimator (Robinson (1995)) and a data-dependent bandwidth parameter. Specifically, letting $I_{n,j}$ be the periodogram ordinate of $\mathbf{X}^n$ at the Fourier frequency $\zeta_j := 2\pi j/n$ $(j = 1, \ldots, \lfloor (n-1)/2 \rfloor)$, the bandwidth is set equal to $\lfloor \{16(-2.2\,\hat{c})^2\}^{-1/5} n^{4/5} \rfloor$, where $\hat{c}$ is the least-squares estimate of the coefficient on $\zeta_j^2/2$ in a pseudo-regression of $\log I_{n,j}$ on $(1, -2\log\zeta_j, \zeta_j^2/2)$, $1 \le j \le \lfloor 0.3 n^{8/9} \rfloor$, and $\lfloor \cdot \rfloor$ denotes the floor function (cf. Henry and Robinson (1996), Andrews and Sun (2004)).

The marginal distributions of both RV and MRV are right-skewed and leptokurtic across all

Table 2: ARSB P-values and Estimated Memory Parameter (RV)

Series	$\hat{\beta}_{1,n}$	$\hat{\beta}_{2,n}$	S_n	K_n	G_n	$\hat{d}$	$SE(\hat{d})$
				RV			
AMGN	6.918	109.98	0.174	0.000	0.000	0.172	0.048
CAT	1.191	5.310	0.000	0.000	0.000	0.397	0.035
GS	1.246	5.237	0.000	0.000	0.000	0.346	0.050
HON	1.792	8.574	0.000	0.000	0.000	0.365	0.048
JNJ	1.860	9.249	0.000	0.000	0.000	0.336	0.046
MMM	2.799	19.457	0.002	0.000	0.000	0.305	0.058
MRK	2.477	14.988	0.000	0.000	0.000	0.243	0.050
TRV	1.809	9.317	0.000	0.000	0.000	0.347	0.043
UNH	1.992	10.480	0.000	0.000	0.000	0.301	0.042
WMT	11.751	261.71	0.293	0.000	0.000	0.284	0.048
				Logarithmic RV			
AMGN	1.017	6.530	0.001	0.000	0.000	0.256	0.042
CAT	0.268	2.930	0.026	0.677	0.031	0.367	0.049
GS	0.292	2.993	0.012	0.964	0.044	0.384	0.052
HON	0.528	3.444	0.002	0.014	0.001	0.347	0.027
JNJ	0.533	3.730	0.000	0.001	0.000	0.353	0.031
MMM	0.548	4.313	0.009	0.000	0.000	0.363	0.057
MRK	0.708	4.265	0.001	0.000	0.000	0.316	0.042
TRV	0.470	3.737	0.003	0.002	0.000	0.312	0.050
UNH	0.493	3.644	0.014	0.002	0.001	0.250	0.017
WMT	0.720	6.028	0.110	0.000	0.000	0.399	0.039

stocks. Furthermore, the bootstrap P-values indicate that the evidence against symmetry, mesokurtosis and normality is very strong in nearly every case. The only exceptions are the symmetry tests for RV for AMGN and WMT, where P-values exceed 0.17, despite the corresponding empirical coefficients of skewness being relatively large.

The logarithmic transformation of RV and MRV substantially reduces both skewness and kurtosis. Although symmetry is rejected at conventional significance levels for all but one RV series and all MRV series, the hypothesis of mesokurtosis cannot be rejected for two RV series (CAT and GS) and four MRV series (CAT, CS, UNH and WMT). However, normality of the marginal distributions of logarithmic RV and MRV is rejected at the 5% significance level in all cases (ARSB P-values range from 0.01 to 0.045 in those cases where evidence against mesokurtosis is weak). These results are consistent with empirical findings reported in the literature. They also align with results obtained from ARSB-assisted tests based on non-studentized statistics (see Section 3.2), reported in Tables 39–40 in the Supplementary Material; the only notable difference is the somewhat weaker evidence against normality of logarithmic MRV for GS obtained using the induced normality test.

Table 3: ARSB P-values and Estimated Memory Parameter (MRV)

Series	$\hat{\beta}_{1,n}$	$\hat{\beta}_{2,n}$	S_n	K_n	G_n	$\hat{d}$	$SE(\hat{d})$
				MRV			
AMGN	11.302	240.06	0.000	0.000	0.000	0.181	0.043
CAT	1.147	4.845	0.000	0.000	0.000	0.384	0.030
GS	1.096	4.372	0.000	0.000	0.000	0.391	0.042
HON	1.932	11.286	0.012	0.000	0.000	0.356	0.051
JNJ	1.616	7.511	0.000	0.000	0.000	0.340	0.039
MMM	2.997	22.537	0.005	0.000	0.000	0.326	0.043
MRK	2.345	15.158	0.001	0.000	0.000	0.269	0.047
TRV	1.627	8.051	0.000	0.000	0.000	0.302	0.047
UNH	1.282	5.336	0.000	0.000	0.000	0.352	0.044
WMT	1.362	5.617	0.000	0.000	0.000	0.387	0.046
				Logarithmic MRV			
AMGN	1.056	8.172	0.046	0.000	0.000	0.259	0.050
CAT	0.251	2.890	0.032	0.578	0.045	0.350	0.051
GS	0.256	2.776	0.029	0.199	0.032	0.401	0.035
HON	0.482	3.477	0.005	0.010	0.001	0.382	0.049
JNJ	0.475	3.433	0.001	0.005	0.001	0.346	0.040
MMM	0.597	4.347	0.022	0.000	0.000	0.372	0.051
MRK	0.559	4.048	0.004	0.000	0.000	0.310	0.043
TRV	0.413	3.548	0.004	0.004	0.000	0.346	0.053
UNH	0.309	3.009	0.018	0.954	0.017	0.399	0.036
WMT	0.389	2.955	0.008	0.817	0.010	0.396	0.047

Lastly, the estimated value of the memory parameter is significantly different from zero for all RV and MRV series, supporting the hypothesis that realized measures of volatility exhibit long memory. The logarithmic transformation generally increases the strength of dependence in both volatility measures, and in some cases $\hat{d}$ is close to the "typical" value of 0.4 reported in the literature. These findings lend support to modeling approaches based on FARIMA models for RV (see, e.g., Andersen et al. (2001), Andersen et al. (2003), Koopman et al. (2005)), although the assumption of Gaussian driving noise may be inappropriate given the evidence against normality of the marginal distribution.

6. Summary

This paper has considered the problem of testing for symmetry, mesokurtosis and normality of the one-dimensional marginal distribution of a strictly stationary fractional process using test statistics based on empirical third-order and fourth-order central moments. Since the asymptotic distributions of the empirical coefficients of skewness and kurtosis, as well as the associated scaling factors, depend on the unknown value of the memory parameter, we have

explored how a suitable bootstrap procedure may be used to facilitate statistical inference in practice. Specifically, we have proposed using the ARSB to studentize the empirical coefficients of skewness and kurtosis and to implement hypothesis tests based on the resulting studentized statistics. As an alternative, we have also considered bootstrap inference based on non-studentized statistics. Both procedures are straightforward to implement and, importantly, do not require prior knowledge or estimation of the memory parameter or the appropriate scaling factors. A simulation study has demonstrated that ARSB-assisted tests for symmetry, mesokurtosis and normality perform well, in terms of level accuracy and power, in the presence of short memory, antipersistence or long memory. An analysis of realized equity-return volatilities for a set of individual stocks has illustrated the practical use of the proposed test procedures.

References

ANDERSEN, T. G., T. BOLLERSLEV, F. X. DIEBOLD, AND H. EBENS (2001): "The Distribution of Realized Stock Return Volatility," Journal of Financial Economics, 61, 43–76.

ANDERSEN, T. G., T. BOLLERSLEV, F. X. DIEBOLD, AND P. LABYS (2003): "Modeling and Forecasting Realized Volatility," Econometrica, 71, 579–625.

ANDERSEN, T. G., D. DOBREV, AND E. SCHAUMBURG (2012): "Jump-Robust Volatility Estimation Using Nearest Neighbor Truncation," Journal of Econometrics, 169, 75–93.

ANDREWS, D. W. K. AND Y. SUN (2004): "Adaptive local polynomial Whittle estimation of long-range dependence," Econometrica, 72, 569–614.

BAI, J. AND S. NG (2005): "Tests for skewness, kurtosis, and normality for time series data," Journal of Business and Economic Statistics, 23, 49–60.

BERAN, J. AND S. GHOSH (1991): "Slowly decaying correlations, testing normality, nuisance parameters," Journal of the American Statistical Association, 86, 785–791.

BICKEL, P. J. AND D. A. FREEDMAN (1981): "Some asymptotic theory for the bootstrap," Annals of Statistics, 9, 1196–1217.

BONDON, P. AND W. PALMA (2007): "A class of antipersistent processes," Journal of Time Series Analysis, 28, 261–273.

BONTEMPS, C. AND N. MEDDAHI (2005): "Testing normality: a GMM approach," Journal of Econometrics, 124, 149–186.

BOUTAHAR, M. (2010): "Behaviour of skewness, kurtosis and normality tests in long memory data," Statistical Methods and Applications, 19, 193–215.

BOWMAN, K. O. AND L. R. SHENTON (1975): "Omnibus test contours for departures from normality based on $\sqrt{b_1}$ and b_2," Biometrika, 62, 243–250.

BÜHLMANN, P. (1997): "Sieve bootstrap for time series," Bernoulli, 3, 123–148.

CASSART, D., M. HALLIN, AND D. PAINDAVEINE (2011): "A class of optimal tests for symmetry based on local Edgeworth approximations," Bernoulli, 17, 1063–1094.

CHENG, Q. (1999): "On time-reversibility of linear processes," Biometrika, 86, 483–486.

CHOW, Y. S. AND H. TEICHER (1997): Probability Theory: Independence, Interchangeability, Martingales, New York: 3rd Edition, Springer-Verlag.

CIPOLLINI, F., G. CRUCIANI, G. M. GALLO, A. INSANA, E. OTRANTO, AND F. SPAGNOLO (2026): "Volatility Archive for Realized Estimates (VOLARE)," ArXiv: 2602.19732v1 [q-fin.ST].

GASSER, T. (1975): "Goodness-of-fit tests for correlated data," Biometrika, 62, 563–570.

GOURIEROUX, C. AND J. JASIAK (2010): "Value at Risk," in Handbook of Financial Econometrics, Vol. 1, ed. by Y. Aït-Sahalia and L. P. Hansen, Amsterdam: North-Holland, 553–615.

HALL, P. (1986): "On the number of bootstrap simulations required to construct a confidence interval," Annals of Statistics, 14, 1453–1462.

HENRY, M. AND P. M. ROBINSON (1996): "Bandwidth choice in Gaussian semiparametric estimation of long range dependence," in Athens Conference on Applied Probability and Time Series Analysis, Vol. II: Time Series Analysis in Memory of E. J. Hannan, ed. by P. M. Robinson and M. Rosenblatt, New York: Springer-Verlag, 220–232.

HO, H.-C. (2002): "On functionals of linear processes with estimated parameters," Statistica Sinica, 12, 1171–1190.

INGERSOLL, J. E. (1987): Theory of Financial Decision Making, Lanham, Maryland: Rowman and Littlefield Publishers.

JARQUE, C. M. AND A. K. BERA (1987): "A test for normality of observations and regression residuals," International Statistical Review, 55, 163–172.

JÖCKEL, K.-H. (1986): "Finite sample properties and asymptotic efficiency of Monte Carlo tests," Annals of Statistics, 14, 336–347.

KAPETANIOS, G. AND Z. PSARADAKIS (2006): "Sieve bootstrap for strongly dependent stationary processes," Working Paper No. 552, Department of Economics, Queen Mary, University of London.

KIEFER, N. M. AND M. SALMON (1983): "Testing normality in econometric models," Economics Letters, 11, 123–127.

KILIAN, L. AND U. DEMIROGLU (2000): "Residual-based tests for normality in autoregressions: asymptotic theory and simulation evidence," Journal of Business and Economic Statistics, 18, 40–50.

KOOPMAN, S. J., B. JUNGBACKER, AND E. HOL (2005): "Forecasting Daily Variability of the S&P 100 Stock Index Using Historical, Realised and Implied Volatility Measurements," Journal of Empirical Finance, 12, 445–475.

KREISS, J.-P. (1992): "Bootstrap procedures for AR(∞) processes," in Bootstrapping and Related Techniques, ed. by K.-H. Jöckel, G. Rothe, and W. Sendler, Heidelberg: Springer-Verlag, 107–113.

LIU, L. Y., A. J. PATTON, AND K. SHEPPARD (2015): "Does Anything Beat 5-minute RV? A Comparison of Realized Measures Across Multiple Asset Classes," Journal of Econometrics, 187, 293–311.

LOBATO, I. N. AND C. VELASCO (2004): "A simple test of normality for time series," Econometric Theory, 20, 671–689.

LOMNICKI, Z. A. (1961): "Tests for departure from normality in the case of linear stochastic processes," Metrika, 4, 37–62.

PENG-ZHOU, J. AND X. SONG (2024): "Self-normalized tests for skewness, kurtosis, and normality for time series data," Available at https://ssrn.com/abstract=4854381.

PIPIRAS, V. AND M. S. TAQQU (2017): Long-Range Dependence and Self-Similarity, Cambridge, UK: Cambridge University Press.

POSKITT, D. S. (2007): "Autoregressive approximation in nonstandard situations: the fractionally integrated and non-invertible cases," Annals of the Institute of Statistical Mathematics, 59, 697–725.

——— (2008): "Properties of the sieve bootstrap for fractionally integrated and non-invertible processes," Journal of Time Series Analysis, 29, 224–250.

PSARADAKIS, Z. (2016): "Using the bootstrap to test for symmetry under unknown dependence," Journal of Business and Economic Statistics, 34, 406–415.

PSARADAKIS, Z. AND M. VÁVRA (2017): "A distance test of normality for a wide class of stationary processes," Econometrics and Statistics, 2, 50–60.

——— (2019): "Bootstrap-assisted tests of symmetry for dependent data," Journal of Statistical Computation and Simulation, 89, 1203–1226.

——— (2020): "Normality tests for dependent data: large-sample and bootstrap approaches," Communications in Statistics–Simulation and Computation, 49, 283–304.

——— (2022): "Using triples to assess symmetry under weak dependence," Journal of Business and Economic Statistics, 40, 1538—1551.

RAMSEY, J. B. AND P. ROTHMAN (1996): "Time irreversibility and business cycle asymmetry," Journal of Money, Credit and Banking, 28, 1–21.

ROBINSON, P. M. (1995): "Gaussian semiparametric estimation of long range dependence," Annals of Statistics, 23, 1630–1661.

SHIBATA, R. (1980): "Asymptotically efficient selection of the order of the model for estimating parameters of a linear process," Annals of Statistics, 8, 147–164.

SICHEL, D. E. (1993): "Business cycle asymmetry: a deeper look," Economic Inquiry, 31, 224–236.

Supplementary Material

A. Skewness and Kurtosis of X_0

Table 1: Coefficient of Skewness of X_0

	d	N	S1	S2	S3	A1	A2	A3	A4
	-0.4	0.0	0.0	0.0	0.7	1.4	1.9	2.9	3.6
M1	0.0	0.0	0.0	0.0	0.0	1.1	1.5	2.3	2.9
	0.4	0.0	0.0	0.0	0.0	0.4	0.6	0.9	1.1
	-0.4	0.0	0.0	0.0	0.0	1.2	1.6	2.5	3.1
M2	0.0	0.0	0.0	0.0	0.0	1.2	1.6	2.6	3.1
	0.4	0.0	0.0	0.0	0.0	0.6	0.8	1.2	1.5
	-0.4	0.0	0.0	0.0	0.0	1.1	1.5	2.4	2.9
M3	0.0	0.0	0.0	0.0	0.0	0.9	1.1	1.8	2.2
	0.4	0.0	0.0	0.0	0.0	0.4	0.5	0.8	0.9
	-0.4	0.0	0.0	0.0	0.0	0.6	0.8	1.3	1.6
M4	0.0	0.0	0.0	0.0	0.0	1.4	1.9	3.0	3.6
	0.4	0.0	0.0	0.0	0.0	1.0	1.3	2.0	2.5
	-0.4	0.0	0.0	0.0	0.0	0.9	1.2	1.9	2.3
M5	0.0	0.0	0.0	0.0	0.0	1.4	1.9	3.0	3.7
	0.4	0.0	0.0	0.0	0.0	1.0	1.3	2.1	2.5
	-0.4	0.0	0.0	0.0	0.0	1.0	1.4	2.2	2.8
M6	0.0	0.0	0.0	0.0	0.0	1.6	2.1	3.4	4.3
	0.4	0.0	0.0	0.0	0.0	1.1	1.5	2.3	3.0

Table 2: Coefficient of Kurtosis of X_0

	d	N	S1	S2	S3	A1	A2	A3	A4
	-0.4	3.0	5.7	11.3	315.0	7.0	20.3	22.1	36.0
M1	0.0	3.0	4.8	8.2	18.6	5.7	14.2	15.3	25.3
	0.4	3.0	3.3	3.9	5.7	3.5	4.9	5.2	6.8
	-0.4	3.0	5.3	9.7	21.7	6.5	19.9	19.1	30.9
M2	0.0	3.0	5.0	8.9	21.5	6.0	15.2	17.5	27.9
	0.4	3.0	3.6	4.6	8.8	3.8	6.2	6.8	10.1
	-0.4	3.0	4.8	8.2	19.6	5.8	14.0	15.8	25.7
M3	0.0	3.0	4.0	5.9	12.5	4.5	8.7	10.1	15.5
	0.4	3.0	3.2	3.6	5.4	3.3	4.3	4.6	5.8
	-0.4	4.3	6.8	11.6	24.8	7.8	17.3	21.2	31.9
M4	0.0	3.9	7.3	13.5	32.6	8.9	22.3	26.6	42.9
	0.4	3.3	5.2	8.6	20.5	6.3	13.3	15.8	25.3
	-0.4	3.2	5.3	9.3	26.0	6.0	16.5	16.8	31.3
M5	0.0	3.0	5.7	11.4	24.9	7.1	24.5	30.7	38.2
	0.4	3.2	4.4	6.7	13.7	5.7	10.8	15.2	21.1
	-0.4	3.2	5.8	11.5	27.5	6.7	19.2	21.6	42.1
M6	0.0	3.1	6.4	13.8	70.0	8.1	24.1	30.0	57.5
	0.4	3.2	4.6	7.2	15.3	6.1	14.1	17.6	32.9

B. Simulation Results for Studentized Statistics

B.1. Model M1

Table 3: Rejection Frequencies of S_n Under M1

		Level				Power			
	d	N	S1	S2	S3	A1	A2	A3	A4
$n = 100$	−0.4	0.052	0.059	0.060	0.055	0.743	0.325	0.731	0.684
	−0.2	0.048	0.063	0.063	0.058	0.650	0.289	0.668	0.626
	0.0	0.040	0.066	0.048	0.060	0.454	0.186	0.542	0.521
	0.2	0.049	0.052	0.061	0.051	0.201	0.110	0.266	0.257
	0.4	0.044	0.049	0.056	0.065	0.088	0.067	0.083	0.115
$n = 200$	−0.4	0.051	0.052	0.048	0.049	0.924	0.521	0.854	0.812
	−0.2	0.058	0.053	0.067	0.046	0.900	0.485	0.839	0.797
	0.0	0.053	0.042	0.042	0.048	0.775	0.353	0.770	0.734
	0.2	0.048	0.049	0.056	0.044	0.382	0.200	0.476	0.460
	0.4	0.047	0.042	0.037	0.040	0.098	0.091	0.149	0.145
$n = 400$	−0.4	0.047	0.054	0.050	0.058	0.975	0.744	0.949	0.910
	−0.2	0.039	0.049	0.041	0.050	0.982	0.690	0.925	0.899
	0.0	0.050	0.048	0.041	0.035	0.943	0.598	0.915	0.864
	0.2	0.040	0.042	0.045	0.039	0.705	0.379	0.784	0.744
	0.4	0.046	0.035	0.051	0.043	0.159	0.114	0.228	0.239

Table 4: Rejection Frequencies of K_n Under M1

		Level	Power						
	d	N	S1	S2	S3	A1	A2	A3	A4
$n = 100$	−0.4	0.054	0.582	0.816	0.889	0.744	0.881	0.964	0.983
	−0.2	0.043	0.580	0.743	0.853	0.680	0.784	0.962	0.971
	0.0	0.047	0.423	0.660	0.757	0.562	0.667	0.877	0.905
	0.2	0.056	0.231	0.386	0.495	0.334	0.436	0.628	0.692
	0.4	0.065	0.102	0.159	0.256	0.151	0.189	0.253	0.336
$n = 200$	−0.4	0.057	0.865	0.972	0.987	0.927	0.979	0.998	1.000
	−0.2	0.056	0.813	0.948	0.985	0.894	0.962	0.997	0.999
	0.0	0.049	0.633	0.883	0.941	0.770	0.905	0.984	0.993
	0.2	0.047	0.344	0.582	0.717	0.448	0.638	0.838	0.889
	0.4	0.039	0.108	0.218	0.307	0.167	0.268	0.368	0.457
$n = 400$	−0.4	0.049	0.981	0.997	1.000	0.996	1.000	1.000	1.000
	−0.2	0.047	0.966	0.999	0.999	0.990	1.000	1.000	1.000
	0.0	0.046	0.871	0.979	0.999	0.943	0.988	1.000	1.000
	0.2	0.050	0.516	0.803	0.889	0.690	0.845	0.978	0.994
	0.4	0.041	0.141	0.282	0.395	0.180	0.350	0.541	0.599

Table 5: Rejection Frequencies of G_n Under M1

		Level	Power						
	d	N	S1	S2	S3	A1	A2	A3	A4
$n = 100$	-0.4	0.049	0.588	0.801	0.876	0.956	0.936	1.000	1.000
	-0.2	0.055	0.578	0.749	0.837	0.927	0.866	1.000	0.999
	0.0	0.052	0.433	0.655	0.767	0.814	0.761	0.991	0.998
	0.2	0.048	0.238	0.404	0.506	0.509	0.526	0.861	0.899
	0.4	0.045	0.119	0.188	0.271	0.202	0.247	0.414	0.475
$n = 200$	-0.4	0.061	0.838	0.957	0.984	1.000	0.991	1.000	1.000
	-0.2	0.046	0.791	0.935	0.985	0.999	0.991	1.000	1.000
	0.0	0.060	0.629	0.864	0.932	0.979	0.943	1.000	1.000
	0.2	0.041	0.350	0.577	0.700	0.749	0.740	0.992	0.996
	0.4	0.045	0.118	0.231	0.313	0.236	0.337	0.561	0.645
$n = 400$	-0.4	0.045	0.977	0.998	1.000	1.000	1.000	1.000	1.000
	-0.2	0.045	0.960	0.997	0.999	1.000	1.000	1.000	1.000
	0.0	0.043	0.848	0.974	0.999	1.000	0.997	1.000	1.000
	0.2	0.038	0.514	0.775	0.880	0.942	0.926	1.000	1.000
	0.4	0.045	0.146	0.307	0.408	0.331	0.439	0.774	0.805

B.2. Model M2

Table 6: Rejection Frequencies of S_n Under M2

		Level				Power			
	d	N	S1	S2	S3	A1	A2	A3	A4
$n = 100$	−0.4	0.056	0.060	0.062	0.058	0.606	0.277	0.600	0.554
	−0.2	0.046	0.066	0.046	0.065	0.625	0.277	0.651	0.600
	0.0	0.052	0.060	0.075	0.068	0.581	0.250	0.612	0.592
	0.2	0.044	0.054	0.054	0.069	0.353	0.170	0.447	0.424
	0.4	0.050	0.061	0.047	0.049	0.126	0.085	0.163	0.144
$n = 200$	−0.4	0.045	0.049	0.046	0.050	0.851	0.450	0.804	0.762
	−0.2	0.045	0.047	0.065	0.045	0.863	0.447	0.833	0.790
	0.0	0.053	0.055	0.064	0.050	0.831	0.425	0.828	0.777
	0.2	0.045	0.053	0.067	0.051	0.617	0.295	0.698	0.693
	0.4	0.047	0.050	0.049	0.036	0.183	0.113	0.290	0.255
$n = 400$	−0.4	0.039	0.056	0.040	0.050	0.967	0.655	0.895	0.861
	−0.2	0.038	0.048	0.034	0.038	0.974	0.694	0.932	0.908
	0.0	0.042	0.059	0.061	0.046	0.965	0.691	0.903	0.885
	0.2	0.052	0.054	0.035	0.043	0.907	0.521	0.866	0.825
	0.4	0.054	0.040	0.037	0.034	0.395	0.204	0.513	0.479

Table 7: Rejection Frequencies of K_n Under M2

		Level	Power						
	d	N	S1	S2	S3	A1	A2	A3	A4
$n = 100$	−0.4	0.049	0.523	0.766	0.850	0.685	0.832	0.948	0.969
	−0.2	0.061	0.551	0.737	0.847	0.667	0.773	0.935	0.954
	0.0	0.042	0.459	0.684	0.808	0.602	0.727	0.908	0.941
	0.2	0.060	0.340	0.496	0.660	0.439	0.579	0.790	0.837
	0.4	0.045	0.193	0.284	0.368	0.227	0.314	0.462	0.556
$n = 200$	−0.4	0.042	0.798	0.932	0.980	0.887	0.965	1.000	0.997
	−0.2	0.043	0.797	0.946	0.986	0.894	0.957	0.998	0.998
	0.0	0.036	0.720	0.909	0.963	0.851	0.907	0.994	0.999
	0.2	0.041	0.522	0.757	0.863	0.678	0.813	0.956	0.966
	0.4	0.054	0.224	0.383	0.506	0.283	0.460	0.652	0.724
$n = 400$	−0.4	0.049	0.949	0.998	1.000	0.994	0.999	1.000	1.000
	−0.2	0.054	0.959	0.998	1.000	0.987	0.999	1.000	1.000
	0.0	0.042	0.930	0.992	0.999	0.973	0.995	1.000	1.000
	0.2	0.065	0.741	0.940	0.986	0.868	0.961	0.994	0.998
	0.4	0.060	0.298	0.522	0.675	0.427	0.628	0.825	0.889

Table 8: Rejection Frequencies of G_n Under M2

		Level	Power						
	d	N	S1	S2	S3	A1	A2	A3	A4
$n = 100$	-0.4	0.047	0.532	0.753	0.835	0.916	0.899	0.999	0.998
	-0.2	0.061	0.551	0.725	0.834	0.918	0.852	1.000	0.999
	0.0	0.050	0.488	0.692	0.800	0.881	0.818	1.000	1.000
	0.2	0.047	0.357	0.512	0.669	0.687	0.680	0.978	0.991
	0.4	0.047	0.207	0.315	0.378	0.370	0.402	0.687	0.744
$n = 200$	-0.4	0.044	0.779	0.923	0.977	0.993	0.979	1.000	1.000
	-0.2	0.050	0.787	0.942	0.970	0.998	0.986	1.000	1.000
	0.0	0.044	0.706	0.889	0.953	0.996	0.962	1.000	1.000
	0.2	0.038	0.518	0.742	0.856	0.935	0.894	1.000	1.000
	0.4	0.052	0.230	0.385	0.504	0.465	0.561	0.858	0.884
$n = 400$	-0.4	0.051	0.944	0.997	0.999	1.000	1.000	1.000	1.000
	-0.2	0.039	0.941	0.996	1.000	1.000	0.999	1.000	1.000
	0.0	0.045	0.923	0.991	0.998	1.000	1.000	1.000	1.000
	0.2	0.057	0.733	0.931	0.984	0.998	0.991	1.000	1.000
	0.4	0.052	0.284	0.511	0.660	0.713	0.727	0.960	0.975

B.3. Model M3

Table 9: Rejection Frequencies of S_n Under M3

		Level				Power			
	d	N	S1	S2	S3	A1	A2	A3	A4
n = 100	−0.4	0.063	0.058	0.060	0.054	0.483	0.222	0.581	0.539
	−0.2	0.051	0.056	0.058	0.078	0.339	0.189	0.450	0.431
	0.0	0.059	0.052	0.074	0.045	0.214	0.124	0.274	0.279
	0.2	0.049	0.038	0.061	0.047	0.100	0.067	0.153	0.165
	0.4	0.042	0.050	0.046	0.054	0.061	0.065	0.082	0.061
n = 200	−0.4	0.051	0.065	0.046	0.053	0.780	0.365	0.777	0.722
	−0.2	0.073	0.060	0.044	0.054	0.614	0.271	0.733	0.648
	0.0	0.070	0.046	0.047	0.060	0.399	0.204	0.520	0.512
	0.2	0.057	0.044	0.054	0.050	0.193	0.121	0.252	0.283
	0.4	0.048	0.033	0.045	0.046	0.075	0.067	0.084	0.093
n = 400	−0.4	0.042	0.049	0.043	0.043	0.958	0.628	0.927	0.874
	−0.2	0.045	0.045	0.044	0.049	0.911	0.524	0.882	0.844
	0.0	0.036	0.039	0.039	0.043	0.718	0.335	0.816	0.761
	0.2	0.043	0.050	0.042	0.046	0.379	0.221	0.524	0.472
	0.4	0.042	0.036	0.049	0.044	0.094	0.087	0.152	0.155

Table 10: Rejection Frequencies of K_n Under M3

		Level	Power						
	d	N	S1	S2	S3	A1	A2	A3	A4
n = 100	−0.4	0.049	0.414	0.653	0.755	0.567	0.730	0.901	0.923
	−0.2	0.051	0.353	0.491	0.653	0.431	0.585	0.774	0.823
	0.0	0.052	0.201	0.388	0.475	0.310	0.405	0.600	0.648
	0.2	0.051	0.122	0.201	0.273	0.162	0.250	0.340	0.401
	0.4	0.045	0.088	0.103	0.129	0.090	0.119	0.159	0.179
n = 200	−0.4	0.057	0.676	0.881	0.938	0.806	0.907	0.993	0.992
	−0.2	0.042	0.527	0.755	0.878	0.653	0.829	0.946	0.975
	0.0	0.063	0.337	0.552	0.693	0.451	0.638	0.824	0.881
	0.2	0.048	0.168	0.310	0.419	0.253	0.377	0.541	0.609
	0.4	0.048	0.083	0.121	0.193	0.102	0.179	0.228	0.302
n = 400	−0.4	0.045	0.904	0.984	0.999	0.965	0.993	1.000	1.000
	−0.2	0.038	0.746	0.946	0.984	0.870	0.965	0.999	0.998
	0.0	0.033	0.493	0.768	0.912	0.673	0.850	0.975	0.984
	0.2	0.051	0.230	0.464	0.630	0.356	0.561	0.770	0.836
	0.4	0.037	0.091	0.169	0.268	0.111	0.226	0.333	0.382

Table 11: Rejection Frequencies of G_n Under M3

		Level	Power						
	d	N	S1	S2	S3	A1	A2	A3	A4
$n = 100$	-0.4	0.054	0.404	0.643	0.739	0.856	0.815	0.997	0.997
	-0.2	0.059	0.356	0.512	0.668	0.692	0.688	0.975	0.980
	0.0	0.049	0.231	0.410	0.464	0.482	0.503	0.860	0.882
	0.2	0.048	0.135	0.236	0.304	0.250	0.324	0.558	0.595
	0.4	0.042	0.099	0.115	0.162	0.118	0.157	0.229	0.266
$n = 200$	-0.4	0.056	0.672	0.853	0.925	0.982	0.949	1.000	1.000
	-0.2	0.066	0.517	0.754	0.875	0.936	0.892	1.000	1.000
	0.0	0.066	0.354	0.546	0.695	0.755	0.758	0.984	0.997
	0.2	0.049	0.175	0.334	0.433	0.421	0.476	0.801	0.836
	0.4	0.046	0.085	0.128	0.219	0.136	0.214	0.329	0.437
$n = 400$	-0.4	0.040	0.881	0.979	0.996	1.000	0.996	1.000	1.000
	-0.2	0.036	0.727	0.934	0.983	1.000	0.991	1.000	1.000
	0.0	0.040	0.485	0.756	0.903	0.959	0.926	1.000	1.000
	0.2	0.036	0.240	0.458	0.619	0.653	0.700	0.978	0.982
	0.4	0.040	0.103	0.195	0.281	0.188	0.274	0.507	0.569

B.4. Model M4

Table 12: Rejection Frequencies of S_n Under M4

		Level				Power			
	d	N	S1	S2	S3	A1	A2	A3	A4
n = 100	−0.4	0.097	0.093	0.082	0.077	0.152	0.092	0.129	0.117
	−0.2	0.098	0.093	0.088	0.082	0.333	0.177	0.435	0.427
	0.0	0.087	0.086	0.066	0.084	0.502	0.243	0.585	0.583
	0.2	0.064	0.067	0.072	0.068	0.571	0.255	0.646	0.671
	0.4	0.065	0.069	0.062	0.062	0.373	0.173	0.430	0.436
n = 200	−0.4	0.106	0.076	0.066	0.051	0.177	0.104	0.188	0.195
	−0.2	0.097	0.076	0.057	0.066	0.532	0.262	0.603	0.592
	0.0	0.087	0.053	0.064	0.052	0.783	0.340	0.792	0.759
	0.2	0.072	0.068	0.059	0.050	0.800	0.409	0.818	0.761
	0.4	0.067	0.060	0.051	0.056	0.586	0.266	0.617	0.595
n = 400	−0.4	0.099	0.038	0.034	0.022	0.247	0.094	0.240	0.211
	−0.2	0.088	0.065	0.061	0.061	0.817	0.401	0.808	0.758
	0.0	0.076	0.061	0.056	0.040	0.932	0.564	0.843	0.851
	0.2	0.056	0.058	0.060	0.047	0.938	0.581	0.901	0.864
	0.4	0.061	0.049	0.050	0.037	0.819	0.407	0.809	0.747

Table 13: Rejection Frequencies of K_n Under M4

		Power							
	d	N	S1	S2	S3	A1	A2	A3	A4
n = 100	−0.4	0.535	0.805	0.860	0.895	0.849	0.904	0.959	0.969
	−0.2	0.454	0.779	0.862	0.906	0.823	0.875	0.944	0.969
	0.0	0.343	0.785	0.881	0.926	0.820	0.883	0.964	0.971
	0.2	0.295	0.738	0.829	0.888	0.784	0.875	0.942	0.956
	0.4	0.181	0.605	0.719	0.753	0.651	0.758	0.862	0.877
n = 200	−0.4	0.783	0.957	0.986	0.990	0.975	0.994	1.000	1.000
	−0.2	0.674	0.964	0.984	0.995	0.978	0.986	1.000	1.000
	0.0	0.591	0.948	0.986	0.994	0.967	0.990	1.000	1.000
	0.2	0.429	0.935	0.979	0.990	0.966	0.981	0.999	0.998
	0.4	0.240	0.781	0.880	0.921	0.852	0.899	0.975	0.983
n = 400	−0.4	0.957	0.999	1.000	1.000	1.000	1.000	1.000	1.000
	−0.2	0.918	0.998	1.000	1.000	1.000	1.000	1.000	1.000
	0.0	0.820	0.999	1.000	1.000	1.000	1.000	1.000	1.000
	0.2	0.627	0.996	1.000	1.000	1.000	0.999	1.000	1.000
	0.4	0.308	0.911	0.982	0.984	0.955	0.981	0.998	0.998

Table 14: Rejection Frequencies of G_n Under M4

		Power							
	d	N	S1	S2	S3	A1	A2	A3	A4
n = 100	−0.4	0.480	0.785	0.846	0.884	0.870	0.892	0.967	0.977
	−0.2	0.420	0.781	0.857	0.890	0.887	0.889	0.987	0.987
	0.0	0.338	0.775	0.858	0.915	0.912	0.901	0.998	0.997
	0.2	0.302	0.710	0.822	0.886	0.909	0.911	0.995	0.996
	0.4	0.195	0.603	0.704	0.753	0.806	0.793	0.957	0.968
n = 200	−0.4	0.414	0.930	0.973	0.989	0.999	0.993	1.000	1.000
	−0.2	0.414	0.930	0.973	0.989	0.999	0.993	1.000	1.000
	0.0	0.414	0.930	0.973	0.989	0.999	0.993	1.000	1.000
	0.2	0.414	0.930	0.973	0.989	0.999	0.993	1.000	1.000
	0.4	0.414	0.930	0.973	0.989	0.999	0.993	1.000	1.000
n = 400	−0.4	0.933	0.995	1.000	1.000	0.999	1.000	1.000	1.000
	−0.2	0.901	0.997	1.000	1.000	1.000	1.000	1.000	1.000
	0.0	0.771	0.999	1.000	1.000	1.000	1.000	1.000	1.000
	0.2	0.586	0.995	0.999	1.000	1.000	1.000	1.000	1.000
	0.4	0.302	0.896	0.974	0.983	0.994	0.988	1.000	1.000

B.5. Model M5

Table 15: Rejection Frequencies of S_n Under M5

		Level				Power			
	d	N	S1	S2	S3	A1	A2	A3	A4
n = 100	−0.4	0.043	0.055	0.052	0.047	0.389	0.159	0.376	0.367
	−0.2	0.051	0.064	0.052	0.052	0.618	0.271	0.671	0.639
	0.0	0.057	0.066	0.071	0.069	0.762	0.325	0.739	0.738
	0.2	0.061	0.086	0.089	0.084	0.694	0.339	0.774	0.723
	0.4	0.085	0.092	0.054	0.076	0.503	0.251	0.543	0.524
n = 200	−0.4	0.061	0.040	0.045	0.034	0.631	0.265	0.589	0.522
	−0.2	0.044	0.041	0.042	0.050	0.848	0.452	0.838	0.783
	0.0	0.063	0.048	0.054	0.055	0.919	0.555	0.886	0.838
	0.2	0.086	0.068	0.071	0.064	0.926	0.557	0.894	0.861
	0.4	0.120	0.101	0.074	0.078	0.716	0.402	0.758	0.720
n = 400	−0.4	0.083	0.034	0.028	0.039	0.847	0.444	0.781	0.699
	−0.2	0.048	0.044	0.045	0.041	0.968	0.650	0.897	0.882
	0.0	0.060	0.042	0.054	0.045	0.982	0.720	0.943	0.914
	0.2	0.084	0.098	0.068	0.071	0.983	0.796	0.953	0.940
	0.4	0.138	0.103	0.127	0.107	0.917	0.632	0.888	0.859

Table 16: Rejection Frequencies of K_n Under M5

		Power							
	d	N	S1	S2	S3	A1	A2	A3	A4
n = 100	−0.4	0.076	0.515	0.705	0.811	0.612	0.762	0.926	0.953
	−0.2	0.063	0.570	0.753	0.865	0.678	0.815	0.949	0.961
	0.0	0.047	0.594	0.802	0.884	0.707	0.825	0.956	0.970
	0.2	0.070	0.570	0.764	0.858	0.662	0.778	0.947	0.966
	0.4	0.053	0.377	0.523	0.657	0.523	0.629	0.811	0.863
n = 200	−0.4	0.105	0.764	0.910	0.976	0.831	0.928	0.998	0.999
	−0.2	0.074	0.825	0.961	0.981	0.903	0.958	0.999	0.999
	0.0	0.064	0.832	0.953	0.989	0.920	0.967	1.000	1.000
	0.2	0.056	0.779	0.937	0.977	0.908	0.965	0.996	1.000
	0.4	0.093	0.532	0.729	0.856	0.670	0.826	0.952	0.971
n = 400	−0.4	0.143	0.951	0.995	0.997	0.970	0.998	1.000	1.000
	−0.2	0.092	0.971	0.999	1.000	0.990	0.999	1.000	1.000
	0.0	0.074	0.974	1.000	1.000	0.996	1.000	1.000	1.000
	0.2	0.100	0.962	0.994	1.000	0.986	0.998	1.000	1.000
	0.4	0.124	0.717	0.917	0.956	0.872	0.935	0.994	0.999

Table 17: Rejection Frequencies of G_n Under M5

		Power							
	d	N	S1	S2	S3	A1	A2	A3	A4
n = 100	−0.4	0.057	0.502	0.690	0.794	0.827	0.824	0.992	0.998
	−0.2	0.059	0.568	0.757	0.857	0.904	0.871	1.000	0.999
	0.0	0.056	0.569	0.791	0.884	0.940	0.896	1.000	1.000
	0.2	0.077	0.563	0.770	0.857	0.914	0.850	0.998	1.000
	0.4	0.082	0.395	0.526	0.681	0.771	0.715	0.976	0.981
n = 200	−0.4	0.091	0.745	0.909	0.975	0.978	0.965	1.000	1.000
	−0.2	0.079	0.814	0.952	0.972	0.996	0.986	1.000	1.000
	0.0	0.069	0.817	0.952	0.986	0.999	0.990	1.000	1.000
	0.2	0.084	0.761	0.931	0.974	0.999	0.989	1.000	1.000
	0.4	0.118	0.536	0.699	0.844	0.920	0.908	0.999	0.995
n = 400	−0.4	0.152	0.933	0.992	0.997	1.000	1.000	1.000	1.000
	−0.2	0.087	0.956	0.999	1.000	1.000	1.000	1.000	1.000
	0.0	0.083	0.971	0.998	1.000	1.000	1.000	1.000	1.000
	0.2	0.107	0.954	0.993	0.999	1.000	1.000	1.000	1.000
	0.4	0.159	0.687	0.896	0.951	0.985	0.979	0.999	1.000

B.6. Model M6

Table 18: Rejection Frequencies of S_n Under M6

		Level				Power			
	d	N	S1	S2	S3	A1	A2	A3	A4
n = 100	−0.4	0.039	0.065	0.044	0.057	0.466	0.192	0.468	0.475
	−0.2	0.046	0.069	0.056	0.067	0.687	0.294	0.703	0.664
	0.0	0.038	0.072	0.077	0.058	0.781	0.359	0.779	0.733
	0.2	0.067	0.079	0.084	0.092	0.748	0.327	0.773	0.733
	0.4	0.080	0.076	0.088	0.069	0.458	0.255	0.567	0.518
n = 200	−0.4	0.048	0.066	0.050	0.044	0.662	0.293	0.652	0.610
	−0.2	0.055	0.041	0.042	0.049	0.883	0.429	0.844	0.803
	0.0	0.048	0.059	0.057	0.059	0.936	0.569	0.889	0.864
	0.2	0.095	0.075	0.080	0.085	0.941	0.574	0.896	0.871
	0.4	0.095	0.093	0.102	0.092	0.714	0.377	0.788	0.755
n = 400	−0.4	0.052	0.029	0.038	0.045	0.856	0.472	0.785	0.751
	−0.2	0.051	0.037	0.054	0.051	0.972	0.666	0.907	0.904
	0.0	0.052	0.061	0.045	0.046	0.981	0.743	0.946	0.921
	0.2	0.076	0.082	0.095	0.078	0.988	0.808	0.966	0.938
	0.4	0.091	0.101	0.137	0.107	0.890	0.592	0.910	0.879

Table 19: Rejection Frequencies of K_n Under M6

		Power							
	d	N	S1	S2	S3	A1	A2	A3	A4
n = 100	−0.4	0.068	0.520	0.714	0.825	0.642	0.780	0.937	0.959
	−0.2	0.055	0.592	0.780	0.874	0.708	0.821	0.958	0.977
	0.0	0.056	0.657	0.819	0.899	0.771	0.850	0.964	0.984
	0.2	0.063	0.580	0.775	0.875	0.726	0.800	0.963	0.975
	0.4	0.058	0.378	0.569	0.665	0.539	0.633	0.849	0.877
n = 200	−0.4	0.087	0.792	0.936	0.982	0.886	0.951	0.999	1.000
	−0.2	0.054	0.847	0.964	0.994	0.937	0.974	0.999	1.000
	0.0	0.062	0.903	0.979	0.990	0.938	0.985	1.000	1.000
	0.2	0.084	0.829	0.957	0.985	0.922	0.966	1.000	1.000
	0.4	0.085	0.548	0.783	0.845	0.715	0.819	0.954	0.982
n = 400	−0.4	0.118	0.972	0.998	0.999	0.990	1.000	1.000	1.000
	−0.2	0.100	0.986	0.999	1.000	0.997	0.999	1.000	1.000
	0.0	0.094	0.987	1.000	1.000	0.998	0.998	1.000	1.000
	0.2	0.103	0.977	1.000	1.000	0.995	1.000	1.000	1.000
	0.4	0.095	0.750	0.934	0.966	0.898	0.954	0.999	1.000

Table 20: Rejection Frequencies of G_n Under M6

		Power							
	d	N	S1	S2	S3	A1	A2	A3	A4
$n = 100$	-0.4	0.052	0.513	0.702	0.814	0.842	0.842	0.995	0.999
	-0.2	0.048	0.595	0.772	0.871	0.934	0.883	0.998	1.000
	0.0	0.054	0.623	0.807	0.891	0.961	0.927	1.000	1.000
	0.2	0.061	0.565	0.786	0.869	0.944	0.869	1.000	1.000
	0.4	0.067	0.401	0.578	0.683	0.757	0.723	0.973	0.981
$n = 200$	-0.4	0.078	0.761	0.920	0.974	0.990	0.969	1.000	1.000
	-0.2	0.061	0.814	0.958	0.990	1.000	0.991	1.000	1.000
	0.0	0.071	0.894	0.971	0.988	0.999	0.998	1.000	1.000
	0.2	0.102	0.816	0.952	0.983	0.998	0.989	1.000	1.000
	0.4	0.101	0.552	0.778	0.855	0.920	0.878	0.994	0.999
$n = 400$	-0.4	0.110	0.954	0.993	0.998	1.000	1.000	1.000	1.000
	-0.2	0.094	0.979	0.999	1.000	1.000	1.000	1.000	1.000
	0.0	0.088	0.981	1.000	1.000	1.000	1.000	1.000	1.000
	0.2	0.113	0.971	0.998	1.000	1.000	1.000	1.000	1.000
	0.4	0.099	0.737	0.927	0.961	0.982	0.980	1.000	1.000

C. Simulation Results for Non-studentized Statistics

C.1. Model M1

Table 21: Rejection Frequencies of Skewness Test Under M1

		Level				Power			
	d	N	S1	S2	S3	A1	A2	A3	A4
$n = 100$	−0.4	0.055	0.077	0.090	0.102	0.819	0.443	0.871	0.810
	−0.2	0.049	0.075	0.074	0.078	0.759	0.337	0.808	0.801
	0.0	0.054	0.081	0.074	0.088	0.528	0.248	0.657	0.649
	0.2	0.051	0.050	0.057	0.074	0.241	0.145	0.334	0.347
	0.4	0.052	0.051	0.048	0.060	0.094	0.065	0.125	0.126
$n = 200$	−0.4	0.050	0.063	0.066	0.072	0.951	0.581	0.937	0.898
	−0.2	0.050	0.063	0.061	0.073	0.941	0.553	0.889	0.870
	0.0	0.045	0.046	0.062	0.059	0.840	0.416	0.838	0.812
	0.2	0.046	0.059	0.057	0.057	0.431	0.220	0.586	0.545
	0.4	0.064	0.040	0.040	0.049	0.125	0.090	0.161	0.156
$n = 400$	−0.4	0.045	0.067	0.053	0.055	0.992	0.789	0.963	0.943
	−0.2	0.049	0.064	0.058	0.043	0.986	0.748	0.950	0.917
	0.0	0.047	0.047	0.056	0.052	0.971	0.640	0.945	0.900
	0.2	0.053	0.056	0.050	0.052	0.749	0.376	0.829	0.782
	0.4	0.051	0.046	0.041	0.039	0.198	0.139	0.294	0.284

Table 22: Rejection Frequencies of Kurtosis Test Under M1

		Level	Power						
	d	N	S1	S2	S3	A1	A2	A3	A4
$n = 100$	−0.4	0.044	0.626	0.806	0.900	0.756	0.851	0.972	0.979
	−0.2	0.056	0.586	0.786	0.862	0.693	0.832	0.953	0.957
	0.0	0.042	0.465	0.639	0.767	0.550	0.678	0.888	0.909
	0.2	0.047	0.228	0.397	0.521	0.327	0.434	0.634	0.694
	0.4	0.053	0.118	0.168	0.253	0.146	0.173	0.302	0.330
$n = 200$	−0.4	0.046	0.851	0.964	0.991	0.938	0.983	0.998	1.000
	−0.2	0.049	0.825	0.949	0.983	0.905	0.963	0.998	1.000
	0.0	0.039	0.676	0.857	0.939	0.782	0.889	0.990	0.991
	0.2	0.045	0.361	0.586	0.719	0.478	0.641	0.862	0.904
	0.4	0.050	0.128	0.225	0.293	0.170	0.256	0.381	0.462
$n = 400$	−0.4	0.061	0.988	1.000	1.000	0.996	0.999	1.000	1.000
	−0.2	0.049	0.977	0.998	1.000	0.993	0.999	1.000	1.000
	0.0	0.040	0.899	0.989	0.996	0.967	0.982	1.000	1.000
	0.2	0.055	0.511	0.796	0.896	0.694	0.872	0.978	0.992
	0.4	0.045	0.158	0.319	0.413	0.197	0.375	0.557	0.629

Table 23: Rejection Frequencies of Normality Test Under M1

		Level	Power						
	d	N	S1	S2	S3	A1	A2	A3	A4
$n = 100$	-0.4	0.058	0.593	0.767	0.877	0.938	0.874	1.000	1.000
	-0.2	0.052	0.538	0.741	0.828	0.862	0.828	0.998	0.998
	0.0	0.044	0.428	0.585	0.716	0.624	0.656	0.938	0.948
	0.2	0.037	0.197	0.363	0.474	0.330	0.417	0.632	0.682
	0.4	0.047	0.099	0.143	0.217	0.126	0.151	0.283	0.298
$n = 200$	-0.4	0.046	0.820	0.950	0.989	0.999	0.982	1.000	1.000
	-0.2	0.040	0.782	0.934	0.981	0.997	0.971	1.000	1.000
	0.0	0.042	0.622	0.825	0.928	0.916	0.870	0.999	0.999
	0.2	0.041	0.300	0.524	0.671	0.513	0.613	0.869	0.906
	0.4	0.041	0.105	0.166	0.266	0.151	0.236	0.345	0.431
$n = 400$	-0.4	0.048	0.982	1.000	1.000	1.000	1.000	1.000	1.000
	-0.2	0.047	0.958	0.998	1.000	1.000	0.999	1.000	1.000
	0.0	0.044	0.862	0.979	0.992	0.996	0.987	1.000	1.000
	0.2	0.051	0.460	0.755	0.882	0.777	0.851	0.992	0.995
	0.4	0.038	0.129	0.261	0.371	0.189	0.337	0.524	0.585

C.2. Model M2

Table 24: Rejection Frequencies of Skewness Test Under M2

		Level				Power			
	d	N	S1	S2	S3	A1	A2	A3	A4
$n = 100$	−0.4	0.057	0.095	0.095	0.096	0.729	0.364	0.825	0.786
	−0.2	0.068	0.077	0.084	0.104	0.734	0.387	0.822	0.813
	0.0	0.044	0.085	0.091	0.084	0.647	0.320	0.801	0.778
	0.2	0.051	0.066	0.079	0.080	0.432	0.211	0.610	0.598
	0.4	0.066	0.057	0.058	0.061	0.176	0.125	0.225	0.232
$n = 200$	−0.4	0.054	0.053	0.063	0.063	0.927	0.552	0.909	0.903
	−0.2	0.047	0.051	0.077	0.091	0.945	0.568	0.924	0.885
	0.0	0.048	0.070	0.070	0.075	0.915	0.513	0.909	0.889
	0.2	0.041	0.068	0.069	0.065	0.704	0.353	0.810	0.772
	0.4	0.052	0.046	0.050	0.050	0.252	0.148	0.342	0.341
$n = 400$	−0.4	0.066	0.051	0.053	0.059	0.984	0.750	0.955	0.923
	−0.2	0.052	0.056	0.071	0.062	0.990	0.793	0.964	0.933
	0.0	0.055	0.058	0.042	0.059	0.982	0.718	0.946	0.918
	0.2	0.049	0.046	0.053	0.046	0.934	0.555	0.921	0.869
	0.4	0.058	0.051	0.050	0.041	0.390	0.209	0.534	0.547

Table 25: Rejection Frequencies of Kurtosis Test Under M2

		Level	Power						
	d	N	S1	S2	S3	A1	A2	A3	A4
$n = 100$	−0.4	0.059	0.559	0.766	0.862	0.685	0.809	0.955	0.978
	−0.2	0.051	0.561	0.743	0.861	0.691	0.799	0.947	0.965
	0.0	0.059	0.485	0.668	0.805	0.632	0.720	0.926	0.943
	0.2	0.037	0.366	0.519	0.665	0.451	0.586	0.795	0.857
	0.4	0.054	0.187	0.281	0.403	0.236	0.387	0.495	0.557
$n = 200$	−0.4	0.037	0.811	0.947	0.986	0.899	0.972	0.998	0.999
	−0.2	0.034	0.799	0.931	0.989	0.902	0.962	0.998	0.999
	0.0	0.038	0.731	0.924	0.971	0.862	0.927	0.998	0.999
	0.2	0.045	0.547	0.753	0.870	0.666	0.819	0.958	0.974
	0.4	0.058	0.223	0.412	0.515	0.327	0.475	0.683	0.743
$n = 400$	−0.4	0.050	0.971	0.999	1.000	0.984	0.999	1.000	1.000
	−0.2	0.044	0.961	0.998	0.999	0.990	0.997	1.000	1.000
	0.0	0.058	0.934	0.995	0.999	0.981	0.996	1.000	1.000
	0.2	0.057	0.754	0.942	0.980	0.893	0.954	0.996	1.000
	0.4	0.045	0.286	0.552	0.733	0.435	0.624	0.842	0.885

Table 26: Rejection Frequencies of Normality Test Under M2

		Level	Power						
	d	N	S1	S2	S3	A1	A2	A3	A4
$n = 100$	-0.4	0.064	0.529	0.718	0.827	0.857	0.806	0.996	0.999
	-0.2	0.058	0.510	0.696	0.829	0.842	0.802	0.995	0.997
	0.0	0.044	0.444	0.625	0.776	0.750	0.707	0.984	0.993
	0.2	0.037	0.311	0.479	0.606	0.513	0.562	0.837	0.880
	0.4	0.052	0.161	0.244	0.357	0.236	0.353	0.467	0.539
$n = 200$	-0.4	0.044	0.750	0.934	0.979	0.988	0.971	1.000	1.000
	-0.2	0.047	0.756	0.908	0.979	0.991	0.966	1.000	1.000
	0.0	0.035	0.690	0.899	0.951	0.969	0.930	1.000	1.000
	0.2	0.042	0.484	0.712	0.836	0.764	0.824	0.986	0.993
	0.4	0.045	0.183	0.360	0.473	0.322	0.433	0.654	0.721
$n = 400$	-0.4	0.062	0.956	0.998	0.999	1.000	1.000	1.000	1.000
	-0.2	0.057	0.949	0.997	0.999	1.000	0.999	1.000	1.000
	0.0	0.046	0.905	0.992	0.999	1.000	0.997	1.000	1.000
	0.2	0.052	0.697	0.910	0.970	0.971	0.947	0.999	1.000
	0.4	0.041	0.240	0.508	0.691	0.457	0.586	0.829	0.886

C.3. Model M3

Table 27: Rejection Frequencies of Skewness Test Under M3

		Level				Power			
	d	N	S1	S2	S3	A1	A2	A3	A4
$n = 100$	−0.4	0.055	0.075	0.076	0.079	0.605	0.287	0.714	0.696
	−0.2	0.046	0.066	0.074	0.087	0.412	0.229	0.584	0.566
	0.0	0.049	0.073	0.073	0.102	0.244	0.176	0.404	0.385
	0.2	0.052	0.073	0.068	0.066	0.133	0.103	0.205	0.180
	0.4	0.062	0.052	0.051	0.061	0.087	0.055	0.071	0.088
$n = 200$	−0.4	0.051	0.062	0.062	0.090	0.850	0.465	0.885	0.843
	−0.2	0.058	0.058	0.066	0.070	0.689	0.378	0.830	0.777
	0.0	0.047	0.057	0.060	0.064	0.473	0.255	0.622	0.596
	0.2	0.052	0.060	0.046	0.063	0.210	0.157	0.300	0.319
	0.4	0.056	0.052	0.044	0.055	0.090	0.060	0.102	0.100
$n = 400$	−0.4	0.053	0.059	0.057	0.054	0.978	0.708	0.935	0.928
	−0.2	0.060	0.060	0.052	0.056	0.942	0.614	0.928	0.892
	0.0	0.057	0.047	0.058	0.055	0.766	0.439	0.898	0.834
	0.2	0.060	0.054	0.047	0.061	0.393	0.238	0.545	0.550
	0.4	0.049	0.056	0.048	0.044	0.106	0.082	0.192	0.167

Table 28: Rejection Frequencies of Kurtosis Test Under M3

		Level	Power						
	d	N	S1	S2	S3	A1	A2	A3	A4
$n = 100$	−0.4	0.050	0.461	0.667	0.777	0.586	0.717	0.897	0.917
	−0.2	0.057	0.353	0.520	0.660	0.454	0.582	0.796	0.826
	0.0	0.050	0.239	0.366	0.514	0.318	0.433	0.592	0.666
	0.2	0.055	0.134	0.210	0.274	0.172	0.246	0.392	0.403
	0.4	0.066	0.084	0.094	0.159	0.105	0.130	0.176	0.208
$n = 200$	−0.4	0.050	0.668	0.889	0.955	0.804	0.926	0.995	0.998
	−0.2	0.051	0.551	0.779	0.890	0.658	0.811	0.961	0.973
	0.0	0.054	0.357	0.560	0.730	0.487	0.656	0.822	0.871
	0.2	0.046	0.173	0.304	0.452	0.251	0.399	0.562	0.644
	0.4	0.045	0.103	0.144	0.181	0.121	0.151	0.227	0.281
$n = 400$	−0.4	0.043	0.906	0.993	0.999	0.963	0.994	1.000	1.000
	−0.2	0.046	0.781	0.952	0.987	0.903	0.964	0.999	0.999
	0.0	0.050	0.535	0.781	0.909	0.677	0.859	0.968	0.986
	0.2	0.049	0.257	0.457	0.655	0.364	0.605	0.767	0.852
	0.4	0.056	0.097	0.179	0.245	0.129	0.216	0.345	0.419

Table 29: Rejection Frequencies of Normality Test Under M3

		Level	Power						
	d	N	S1	S2	S3	A1	A2	A3	A4
$n = 100$	-0.4	0.047	0.402	0.608	0.736	0.688	0.690	0.966	0.971
	-0.2	0.053	0.299	0.470	0.608	0.484	0.565	0.843	0.862
	0.0	0.043	0.213	0.324	0.478	0.323	0.407	0.592	0.661
	0.2	0.039	0.103	0.181	0.243	0.167	0.228	0.367	0.380
	0.4	0.046	0.056	0.074	0.137	0.088	0.107	0.148	0.176
$n = 200$	-0.4	0.049	0.618	0.861	0.937	0.940	0.924	1.000	1.000
	-0.2	0.047	0.495	0.730	0.868	0.761	0.799	0.987	0.995
	0.0	0.042	0.297	0.514	0.672	0.526	0.621	0.841	0.879
	0.2	0.041	0.141	0.262	0.402	0.239	0.358	0.535	0.629
	0.4	0.034	0.074	0.123	0.150	0.098	0.130	0.203	0.251
$n = 400$	-0.4	0.039	0.882	0.989	0.998	1.000	0.995	1.000	1.000
	-0.2	0.044	0.728	0.939	0.980	0.982	0.966	1.000	1.000
	0.0	0.053	0.460	0.734	0.881	0.774	0.833	0.990	0.995
	0.2	0.045	0.210	0.402	0.610	0.383	0.562	0.769	0.830
	0.4	0.040	0.070	0.141	0.216	0.112	0.195	0.324	0.381

C.4. Model M4

Table 30: Rejection Frequencies of Skewness Test Under M4

		Level				Power			
	d	N	S1	S2	S3	A1	A2	A3	A4
$n = 100$	−0.4	0.118	0.140	0.102	0.097	0.175	0.145	0.201	0.207
	−0.2	0.111	0.126	0.154	0.144	0.421	0.278	0.581	0.538
	0.0	0.090	0.120	0.121	0.105	0.620	0.353	0.808	0.760
	0.2	0.093	0.090	0.101	0.113	0.666	0.324	0.822	0.800
	0.4	0.081	0.084	0.094	0.083	0.458	0.205	0.572	0.554
$n = 200$	−0.4	0.088	0.079	0.071	0.062	0.211	0.108	0.227	0.243
	−0.2	0.099	0.106	0.080	0.082	0.651	0.335	0.734	0.701
	0.0	0.081	0.077	0.098	0.073	0.848	0.443	0.873	0.853
	0.2	0.077	0.086	0.086	0.070	0.876	0.495	0.871	0.884
	0.4	0.077	0.057	0.062	0.066	0.650	0.324	0.704	0.673
$n = 400$	−0.4	0.081	0.043	0.030	0.026	0.263	0.129	0.267	0.269
	−0.2	0.112	0.093	0.079	0.060	0.859	0.466	0.848	0.800
	0.0	0.072	0.064	0.066	0.059	0.952	0.604	0.917	0.886
	0.2	0.065	0.070	0.057	0.056	0.960	0.704	0.941	0.911
	0.4	0.049	0.050	0.051	0.066	0.878	0.492	0.853	0.821

Table 31: Rejection Frequencies of Kurtosis Test Under M4

		Power							
	d	N	S1	S2	S3	A1	A2	A3	A4
$n = 100$	−0.4	0.578	0.816	0.880	0.919	0.846	0.908	0.959	0.978
	−0.2	0.479	0.791	0.889	0.922	0.845	0.916	0.972	0.970
	0.0	0.384	0.799	0.887	0.917	0.838	0.874	0.968	0.967
	0.2	0.296	0.770	0.864	0.915	0.799	0.877	0.953	0.960
	0.4	0.195	0.588	0.709	0.774	0.685	0.729	0.877	0.898
$n = 200$	−0.4	0.782	0.979	0.983	0.992	0.981	0.984	0.998	1.000
	−0.2	0.711	0.959	0.991	0.994	0.976	0.990	0.996	1.000
	0.0	0.569	0.961	0.989	1.000	0.969	0.991	0.999	1.000
	0.2	0.416	0.947	0.978	0.991	0.966	0.985	0.998	0.999
	0.4	0.225	0.760	0.892	0.919	0.863	0.899	0.976	0.978
$n = 400$	−0.4	0.971	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	−0.2	0.916	0.999	1.000	1.000	1.000	1.000	1.000	1.000
	0.0	0.843	0.999	1.000	1.000	1.000	1.000	1.000	1.000
	0.2	0.635	0.997	1.000	1.000	1.000	1.000	1.000	1.000
	0.4	0.283	0.916	0.976	0.989	0.969	0.990	0.998	1.000

Table 32: Rejection Frequencies of Normality Test Under M4

		Power							
	d	N	S1	S2	S3	A1	A2	A3	A4
n = 100	−0.4	0.515	0.760	0.848	0.889	0.807	0.883	0.952	0.971
	−0.2	0.422	0.749	0.855	0.898	0.836	0.904	0.975	0.981
	0.0	0.333	0.752	0.853	0.890	0.881	0.863	0.997	0.992
	0.2	0.255	0.718	0.840	0.886	0.891	0.866	0.994	0.992
	0.4	0.173	0.557	0.673	0.738	0.709	0.704	0.909	0.915
n = 200	−0.4	0.715	0.966	0.980	0.990	0.970	0.980	0.998	1.000
	−0.2	0.647	0.945	0.986	0.993	0.976	0.989	0.999	1.000
	0.0	0.497	0.939	0.984	0.999	0.989	0.990	1.000	1.000
	0.2	0.364	0.929	0.968	0.989	0.997	0.979	1.000	1.000
	0.4	0.199	0.719	0.870	0.901	0.894	0.885	0.983	0.983
n = 400	−0.4	0.950	0.999	0.999	1.000	1.000	1.000	1.000	1.000
	−0.2	0.875	0.998	1.000	1.000	1.000	1.000	1.000	1.000
	0.0	0.795	0.999	1.000	1.000	1.000	1.000	1.000	1.000
	0.2	0.555	0.997	1.000	1.000	1.000	1.000	1.000	1.000
	0.4	0.232	0.892	0.969	0.985	0.984	0.990	0.999	1.000

C.5. Model M5

Table 33: Rejection Frequencies of Skewness Test Under M5

		Level				Power			
	d	N	S1	S2	S3	A1	A2	A3	A4
$n = 100$	−0.4	0.045	0.059	0.069	0.065	0.470	0.227	0.586	0.580
	−0.2	0.057	0.068	0.066	0.084	0.736	0.360	0.843	0.818
	0.0	0.054	0.066	0.074	0.100	0.832	0.416	0.894	0.867
	0.2	0.066	0.075	0.096	0.107	0.817	0.420	0.858	0.842
	0.4	0.100	0.101	0.106	0.094	0.549	0.284	0.677	0.632
$n = 200$	−0.4	0.053	0.060	0.048	0.056	0.756	0.350	0.734	0.669
	−0.2	0.052	0.061	0.059	0.066	0.908	0.552	0.900	0.903
	0.0	0.039	0.064	0.066	0.054	0.949	0.583	0.937	0.918
	0.2	0.080	0.089	0.093	0.096	0.962	0.633	0.949	0.900
	0.4	0.124	0.098	0.104	0.097	0.764	0.458	0.827	0.785
$n = 400$	−0.4	0.068	0.053	0.037	0.036	0.887	0.498	0.792	0.758
	−0.2	0.051	0.038	0.044	0.055	0.983	0.694	0.943	0.924
	0.0	0.041	0.042	0.066	0.060	0.990	0.787	0.967	0.943
	0.2	0.093	0.093	0.096	0.081	0.991	0.808	0.974	0.946
	0.4	0.160	0.115	0.123	0.107	0.932	0.687	0.917	0.886

Table 34: Rejection Frequencies of Kurtosis Test Under M5

		Power							
	d	N	S1	S2	S3	A1	A2	A3	A4
$n = 100$	−0.4	0.077	0.554	0.727	0.820	0.637	0.781	0.926	0.950
	−0.2	0.065	0.583	0.793	0.881	0.711	0.804	0.945	0.973
	0.0	0.067	0.614	0.812	0.913	0.729	0.858	0.963	0.976
	0.2	0.049	0.548	0.756	0.861	0.709	0.810	0.947	0.955
	0.4	0.081	0.372	0.506	0.681	0.512	0.625	0.819	0.876
$n = 200$	−0.4	0.113	0.776	0.922	0.974	0.867	0.940	0.991	0.999
	−0.2	0.080	0.826	0.950	0.985	0.910	0.963	0.999	1.000
	0.0	0.064	0.849	0.967	0.988	0.913	0.975	0.998	0.999
	0.2	0.078	0.796	0.953	0.975	0.894	0.967	0.998	0.997
	0.4	0.102	0.527	0.750	0.856	0.714	0.814	0.972	0.977
$n = 400$	−0.4	0.152	0.958	0.997	0.999	0.983	0.995	1.000	1.000
	−0.2	0.085	0.965	0.999	0.999	0.997	0.999	1.000	1.000
	0.0	0.089	0.973	1.000	1.000	0.996	1.000	1.000	1.000
	0.2	0.102	0.968	0.998	1.000	0.988	1.000	1.000	1.000
	0.4	0.126	0.728	0.910	0.965	0.871	0.945	0.997	0.998

Table 35: Rejection Frequencies of Normality Test Under M5

		Power							
	d	N	S1	S2	S3	A1	A2	A3	A4
n = 100	−0.4	0.069	0.502	0.670	0.780	0.722	0.767	0.974	0.981
	−0.2	0.067	0.544	0.746	0.845	0.895	0.807	0.999	0.999
	0.0	0.063	0.563	0.771	0.888	0.915	0.872	1.000	1.000
	0.2	0.059	0.488	0.714	0.816	0.911	0.819	0.998	1.000
	0.4	0.099	0.325	0.475	0.630	0.630	0.628	0.895	0.920
n = 200	−0.4	0.110	0.735	0.891	0.962	0.950	0.938	1.000	1.000
	−0.2	0.065	0.801	0.928	0.981	0.995	0.979	1.000	1.000
	0.0	0.054	0.804	0.953	0.985	0.995	0.976	1.000	1.000
	0.2	0.072	0.758	0.935	0.971	0.999	0.980	1.000	1.000
	0.4	0.118	0.494	0.717	0.829	0.837	0.823	0.992	0.990
n = 400	−0.4	0.147	0.936	0.994	0.999	1.000	0.997	1.000	1.000
	−0.2	0.083	0.951	0.995	0.999	1.000	1.000	1.000	1.000
	0.0	0.075	0.962	0.999	1.000	1.000	1.000	1.000	1.000
	0.2	0.105	0.954	0.998	1.000	1.000	0.999	1.000	1.000
	0.4	0.156	0.681	0.888	0.954	0.960	0.951	1.000	1.000

C.6. Model M6

Table 36: Rejection Frequencies of Skewness Test Under M6

		Level				Power			
	d	N	S1	S2	S3	A1	A2	A3	A4
$n = 100$	−0.4	0.066	0.078	0.087	0.087	0.563	0.263	0.687	0.634
	−0.2	0.051	0.079	0.081	0.110	0.791	0.409	0.855	0.859
	0.0	0.036	0.092	0.106	0.107	0.878	0.460	0.900	0.900
	0.2	0.078	0.085	0.096	0.130	0.840	0.421	0.898	0.859
	0.4	0.090	0.090	0.103	0.103	0.508	0.277	0.649	0.634
$n = 200$	−0.4	0.050	0.056	0.064	0.058	0.769	0.371	0.761	0.756
	−0.2	0.050	0.074	0.071	0.084	0.942	0.583	0.925	0.908
	0.0	0.052	0.058	0.085	0.094	0.965	0.640	0.952	0.921
	0.2	0.070	0.095	0.102	0.116	0.966	0.653	0.943	0.935
	0.4	0.085	0.124	0.102	0.116	0.759	0.411	0.824	0.813
$n = 400$	−0.4	0.048	0.049	0.052	0.031	0.887	0.538	0.839	0.806
	−0.2	0.040	0.054	0.053	0.057	0.981	0.731	0.960	0.940
	0.0	0.042	0.064	0.064	0.062	0.987	0.809	0.973	0.964
	0.2	0.081	0.090	0.094	0.083	0.987	0.814	0.977	0.967
	0.4	0.119	0.117	0.127	0.124	0.923	0.659	0.937	0.931

Table 37: Rejection Frequencies of Kurtosis Test Under M6

		Power							
	d	N	S1	S2	S3	A1	A2	A3	A4
$n = 100$	−0.4	0.070	0.538	0.744	0.857	0.661	0.809	0.952	0.969
	−0.2	0.047	0.651	0.815	0.886	0.731	0.837	0.971	0.975
	0.0	0.064	0.673	0.807	0.918	0.756	0.852	0.987	0.982
	0.2	0.073	0.601	0.799	0.904	0.736	0.827	0.966	0.982
	0.4	0.078	0.399	0.602	0.679	0.539	0.668	0.876	0.880
$n = 200$	−0.4	0.093	0.828	0.931	0.986	0.901	0.964	0.997	0.998
	−0.2	0.055	0.882	0.974	0.993	0.933	0.979	1.000	1.000
	0.0	0.059	0.899	0.983	0.994	0.949	0.984	0.999	0.999
	0.2	0.079	0.835	0.967	0.989	0.935	0.973	0.999	1.000
	0.4	0.081	0.581	0.741	0.859	0.720	0.829	0.972	0.989
$n = 400$	−0.4	0.124	0.976	0.998	0.999	0.985	0.999	1.000	1.000
	−0.2	0.087	0.985	1.000	1.000	0.994	1.000	1.000	1.000
	0.0	0.068	0.994	1.000	1.000	0.998	1.000	1.000	1.000
	0.2	0.092	0.972	1.000	1.000	0.994	1.000	1.000	1.000
	0.4	0.103	0.767	0.921	0.974	0.905	0.961	1.000	1.000

Table 38: Rejection Frequencies of Normality Test Under M6

		Power							
	d	N	S1	S2	S3	A1	A2	A3	A4
n = 100	−0.4	0.066	0.487	0.690	0.809	0.776	0.795	0.979	0.985
	−0.2	0.045	0.579	0.770	0.863	0.906	0.857	1.000	0.999
	0.0	0.061	0.603	0.773	0.898	0.959	0.870	1.000	1.000
	0.2	0.076	0.544	0.761	0.872	0.913	0.846	1.000	1.000
	0.4	0.079	0.344	0.554	0.624	0.633	0.657	0.924	0.917
n = 200	−0.4	0.084	0.776	0.905	0.981	0.960	0.963	1.000	1.000
	−0.2	0.066	0.841	0.955	0.991	0.999	0.985	1.000	1.000
	0.0	0.058	0.866	0.973	0.990	1.000	0.994	1.000	1.000
	0.2	0.086	0.796	0.950	0.983	0.993	0.977	1.000	1.000
	0.4	0.094	0.526	0.703	0.835	0.814	0.822	0.985	0.992
n = 400	−0.4	0.116	0.962	0.995	0.999	0.999	1.000	1.000	1.000
	−0.2	0.069	0.979	1.000	1.000	1.000	1.000	1.000	1.000
	0.0	0.063	0.989	1.000	1.000	1.000	1.000	1.000	1.000
	0.2	0.101	0.967	1.000	1.000	1.000	1.000	1.000	1.000
	0.4	0.124	0.719	0.910	0.969	0.954	0.961	1.000	1.000

D. Empirical Results: Non-studentized Statistics

Table 39: ARSB P-values and Estimated Memory Parameter (RV)

Series	$\hat{\beta}_{1,n}$	$\hat{\beta}_{2,n}$	Skewness	Kurtosis	Normality	$\hat{d}$	SE($\hat{d}$)
				RV			
AMGN	6.918	109.980	0.084	0.000	0.000	0.172	0.048
CAT	1.191	5.310	0.000	0.000	0.000	0.397	0.035
GS	1.246	5.237	0.000	0.000	0.000	0.346	0.050
HON	1.792	8.574	0.000	0.000	0.000	0.365	0.048
JNJ	1.86	9.249	0.000	0.000	0.000	0.336	0.046
MMM	2.799	19.457	0.001	0.000	0.000	0.305	0.058
MRK	2.477	14.988	0.000	0.000	0.000	0.243	0.050
TRV	1.809	9.317	0.000	0.000	0.000	0.347	0.043
UNH	1.992	10.480	0.000	0.000	0.000	0.301	0.042
WMT	11.751	261.710	0.193	0.000	0.000	0.284	0.048
				Logarithmic RV			
AMGN	1.017	6.530	0.003	0.000	0.000	0.256	0.042
CAT	0.268	2.930	0.021	0.678	0.042	0.367	0.049
GS	0.292	2.993	0.017	0.965	0.033	0.384	0.052
HON	0.528	3.444	0.000	0.012	0.001	0.347	0.027
JNJ	0.533	3.730	0.002	0.000	0.000	0.353	0.031
MMM	0.548	4.313	0.011	0.000	0.000	0.363	0.057
MRK	0.708	4.265	0.000	0.000	0.000	0.316	0.042
TRV	0.470	3.737	0.006	0.000	0.001	0.312	0.050
UNH	0.493	3.644	0.006	0.001	0.002	0.250	0.017
WMT	0.720	6.028	0.115	0.000	0.000	0.399	0.039

Table 40: ARSB P-values and Estimated Memory Parameter (MRV)

Series	$\hat{\beta}_{1,n}$	$\hat{\beta}_{2,n}$	Skewness	Kurtosis	Normality	$\hat{d}$	SE($\hat{d}$)
				MRV			
AMGN	11.302	240.06	0.000	0.000	0.000	0.181	0.043
CAT	1.147	4.845	0.000	0.000	0.000	0.384	0.030
GS	1.096	4.372	0.000	0.000	0.000	0.391	0.042
HON	1.932	11.286	0.004	0.000	0.000	0.356	0.051
JNJ	1.616	7.511	0.000	0.000	0.000	0.340	0.039
MMM	2.997	22.537	0.010	0.000	0.000	0.326	0.043
MRK	2.345	15.158	0.003	0.000	0.000	0.269	0.047
TRV	1.627	8.051	0.000	0.000	0.000	0.302	0.047
UNH	1.282	5.336	0.000	0.000	0.000	0.352	0.044
WMT	1.362	5.617	0.000	0.000	0.000	0.387	0.046
				Logarithmic MRV			
AMGN	1.056	8.172	0.036	0.000	0.000	0.259	0.050
CAT	0.251	2.890	0.023	0.514	0.045	0.350	0.051
GS	0.256	2.776	0.031	0.196	0.061	0.401	0.035
HON	0.482	3.477	0.004	0.010	0.008	0.382	0.049
JNJ	0.475	3.433	0.001	0.010	0.002	0.346	0.040
MMM	0.597	4.347	0.024	0.000	0.000	0.372	0.051
MRK	0.559	4.048	0.004	0.000	0.000	0.310	0.043
TRV	0.413	3.548	0.009	0.003	0.006	0.346	0.053
UNH	0.309	3.009	0.021	0.959	0.041	0.399	0.036
WMT	0.389	2.955	0.018	0.808	0.035	0.396	0.047