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Spatial Synthetic Difference-in-Differences

Renan Serenini[*] Frantisek Masek[†]

Abstract

We propose a spatial extension of the Synthetic Difference-in-Differences (SyDiD) estimator developed by Arkhangelsky et al. (2021). Our estimator addresses violations of the Stable Unit Treatment Value Assumption (SUTVA) that arise when treatment effects spill over to other units. Spillovers to units in the donor pool can lead to biased and inconsistent estimates of the Average Treatment Effect on the Treated (ATT), while spillovers outside the donor pool leave the ATT identifiable but prevent identification of the Average Treatment Effect (ATE). Building on the framework of the Spatial Difference-in-Differences estimator introduced by Delgado and Florax (2015), we develop a method that decomposes the ATE into direct (ATT) and indirect treatment effects. We demonstrate that our estimator improves the identification of the indirect effect relative to the standard Spatial Difference-in-Differences approach, while retaining the robustness and favorable properties of the Synthetic Difference-in-Differences method for estimating the direct effect.

Keywords: Average Treatment Effect, Treatment Spillovers, Synthetic Difference-in-Differences

JEL classification: C21, C23, D62, I18

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Non-technical summary

This paper introduces a new method for estimating causal effects when treatment may affect not only directly treated units but also other units connected to them. The proposed estimator, Spatial Synthetic Difference-in-Differences (SpSyDiD), extends the Synthetic Difference-in-Differences (SyDiD) approach of Arkhangelsky et al. (2021). SyDiD merges the best features of two widely used identification strategies: Synthetic Control Method (SCM) and Difference-in-Differences (DiD). By combining these approaches, SyDiD provides a more robust estimator that is less sensitive to violations of parallel trends and unobserved heterogeneity.

However, SyDiD, like many causal inference methods, relies on the Stable Unit Treatment Value Assumption (SUTVA), which requires that a unit's outcome depends only on its own treatment status. Violations of this assumption can arise when treatment effects spill over to other units. Two scenarios are important to distinguish. In the first, spillovers affect control units, biasing the estimate of the Average Treatment Effect on the Treated (ATT) because the control group no longer reflects untreated outcomes. The usual remedy is to exclude possibly affected units from the donor pool. But doing so precludes the estimation of indirect effects, and thus prevents the researcher from identifying the Average Treatment Effect (ATE). In the second case, spillovers do not reach the control group, so ATT remains unbiased. Yet again, the ATE cannot be identified, since it requires knowledge of both direct and indirect effects.

SpSyDiD addresses the spillovers issue explicitly. It incorporates spatial spillovers into the SyDiD framework using a spatial structure to capture how treatment in one unit can influence others. This allows researchers to disentangle the direct effect on treated units from the indirect (spillover) effect on exposed but untreated units, all while preserving the desirable weighting features of SyDiD. Compared to previous spatial DiD methods, SpSyDiD provides more accurate and efficient estimates, particularly when indirectly affected units are similar to treated ones.

The paper validates the method through simulation studies and applies it to a real-world case: the 2007 Legal Arizona Workers Act. While earlier research estimated only the direct impact of the law, SpSyDiD reveals significant spillover effects on nearby units, suggesting a broader footprint of the intervention than previously documented. The method thus offers a powerful tool for researchers interested in understanding how policies propagate through interconnected settings.

1. Introduction

The Synthetic Difference-in-Differences (SyDiD) estimator of Arkhangelsky et al. (2021) offers an intriguing merge of ideas coming from two widely used tools in the quasi-experimental literature - Synthetic Control Method(s) (SCM) and Difference-in-Differences (DiD). The authors show that using reweighting of control units delivered by the synthetic control component within a two-way fixed setting may be competitive, or possibly superior, in situations where each of the aforementioned approaches would be deployed individually. While the reweighting weakens the reliance on the parallel trend assumptions, unlike in the conventional SCM, SyDiD is also invariant to additive unit-level shifts as it is common in the DiD literature.[1]

To the best of our knowledge, there has not been developed any extension of Arkhangelsky et al. (2021) that would be able to tackle the situation of possible spillovers of the treatment. In such a situation, the Stable Unit Treatment Value Assumption (SUTVA) is violated, and standard quasi-experimental methods will be biased and inconsistent.[2] In DiD, some approaches try to handle this problem by extending the DiD estimators for spatial components (Delgado and Florax, 2015; Butts, 2021). The case when treatment may spill over to other units is studied also in the SCM literature (Cao and Dowd, 2019; Di Stefano and Mellace, 2024). Yet no such counterpart has appeared for the SyDiD estimator.

We come with a spatial extension of SyDiD building on the approach of Delgado and Florax (2015). In their DiD for spatial data, treatment can spill over to other units otherwise considered in the control group. Hence, the potential outcome of observed units depends not only on their own treatment status, but also on treatment of units experiencing spillovers. Delgado and Florax (2015) control for this process by utilizing spatial econometrics tools as the spatial weights' matrix to incorporate treatment

[1]For the sake of clarity, we do not elaborate on other features that make the estimator of Arkhangelsky et al. (2021) appealing as weighting of time periods or using regularization. Although these may improve the properties of the estimator, they are not at the core of our interest. However, they may be used in our spatial extension of SyDiD too.

[2]Note that we focus on one particular violation of the Stable Unit Treatment Value Assumption (SUTVA), as formulated in Rubin (1990) and Angrist et al. (1996). Specifically, we consider the component of SUTVA which requires that the potential outcomes of a given unit are unaffected by the treatment status of other units. We do not address the other component of SUTVA, which requires consistency of the treatment across units (i.e., no hidden variation in the treatment). A violation of either component implies a failure of the SUTVA assumption.

spillovers (Anselin, 1988). Specifically, the spatial interaction related to treatment responses makes their approach isomorphic to the SLX model (Vega and Elhorst, 2015). In our approach, we incorporate Delgado and Florax (2015) into Arkhangelsky et al. (2021) to develop a Spatial SyDiD (SpSyDiD).

Using two crafted simulation studies, we demonstrate that the spatial treatment extension of SyDiD substantially improves the estimation of the Average Treatment Effect (ATE) in the situation of the SUTVA violation. Our estimator can capture both direct and indirect treatment effects, as it is the case for the conventional DiD in the spatial extension of Delgado and Florax (2015). What is more, we do not alter in any way the comparison of the direct treatment effect between SyDiD and DiD. Therefore, the characteristics related to the comparison discussed in Arkhangelsky et al. (2021) also hold in our approach.

A more nuanced discussion is necessary when comparing our estimator to Delgado and Florax (2015) for the case of the indirect treatment effect. Given that we use the same weights stemming from the synthetic control constructed for directly treated units also for the estimation of the indirect treatment effect, we assume a close similarity of directly and indirectly treated units. If this assumption holds and hence the synthetic control constructed for directly treated units is better than the uniform weighting embedded in spatial DiD, our estimator improves on the approach of Delgado and Florax (2015) also in the estimation of the indirect treatment effect. In the simulated studies, we show that this is usually the case.

The units that received the treatment spillover because of interactions with directly treated units must be similar enough to treated units. In such a situation, the weights obtained for directly treated units cause that also indirectly treated units are compared to more similar units rather than to all the units in the donor pool. The more similar units are of a greater importance given that these are similar for both directly and indirectly treated units, unlike in Delgado and Florax (2015) where each unit in the donor pool obtains the same weight.

We present an example of our method using real data. Specifically, we follow Bohn et al. (2014) who estimate the effect of the Arizona's 2007 Legal Arizona Workers Act (LAWA) on the decrease in the proportion of noncitizen Hispanic subpopulation. Since we can estimate the indirect effect of the treatment (spillover) on some of the neighboring states, we are able to observe some movements in the affected subpopulation. In other words, we can estimate the fraction of people who moved from Arizona to neighboring states. We observe statistically significant movements to close states for the working subpopulation.

Related literature. The closest to our estimator is the work of Delgado and Florax (2015). Their spatial DiD can be even considered a special case of our approach in the situation of uniform weights. Using Monte Carlo simulations, they show that incorporating spatial interaction of treatment decreases bias caused by the SUTVA violation when comparing their estimator to a simple DiD setup. Hence, they can estimate both direct and indirect treatment effects. Their method can be perceived as the SLX model coming from the spatial econometrics literature (Vega and Elhorst, 2015). As already mentioned above, we follow up on Delgado and Florax (2015) and extend their method using reweighting of the units in the donor pool.

A different perspective is taken in the work of Butts (2021) who discusses two sources of the bias due to the presence of spillovers.[3] The author mentions the fact that the control units affected by the treatment spillovers cannot serve as counterfactual trend, given that their outcomes are affected by treatment. Moreover, he also points out that changes in the treated units' outcomes may be influenced by the treatment effects of the close units through a sort of general equilibrium forces. He suggests modeling spillovers in a general form using a 'Rings' style estimator. Specifically, Butts (2021) proposes a way of handling both sources of bias while semi-parametrically estimating possible spillovers. Moreover, he embeds the staggered treatment into his method. In the following work, Butts (2023) presents a data-driven ring selection process. The estimator then compares units immediately next to treatment (an inner-ring) to units just slightly further away (an outer-ring).

Another pioneer in the DiD spillovers-broadened literature is Clarke (2017). His estimator considers treatment spillovers using two classes of estimands - besides standard treatment effects, it also works with so-called "close" to treatment effects. The method offers a procedure of defining the distance through which treatment

[3] Although Butts (2021) grasps the problem from a distinct angle, one can show that his approach is isomorphic to the estimator of Delgado and Florax (2015).

propagates, while distance may be defined even as a multidimensional measure.

Conceptually distinct, the DiD estimator is used in Dubé et al. (2014) to estimate the effect of public mass transit systems on real-estate prices. The authors conduct the DiD analysis within the Spatial Autoregressive model (SAR) (Anselin, 1988). Hence, given the global property of the SAR model, their approach allows for general equilibrium feedback effects that one cannot obtain using the method of Delgado and Florax (2015). Kolak and Anselin (2020) discuss their approach in a thorough survey of various attempts extending quasi-experimental methods used in the causal inference literature by spatial aspects. Their article may serve as a coherent summary of many approaches that are developed to handle the SUTVA violation.

Moving to the Synthetic Control Method(s) literature, we stress two articles that consider the spillovers' problem. Di Stefano and Mellace (2024) develop the Inclusive Synthetic Control Method which allows including even indirectly affected units in the donor pool. Given that their method does not need to restrict the donor pool into pure controls and affected units, it can be useful in applications where incorporating even the indirectly treated units is unavoidable to get a reasonable control unit. Di Stefano and Mellace (2024) use the case of the German reunification to show the method's implementation and its comparison to the conventional Synthetic Control Method of Abadie et al. (2015) and the restricted version of SCM from Abadie and L'Hour (2021).

Cao and Dowd (2019) propose method-wise different way of estimating both direct treatment effects and spillover effects. Cao and Dowd (2019) impose a linear assumption on both direct treatment effect and spillover effects. They show that their estimations are asymptotically unbiased, while building even an inferential procedure that is asymptotically unbiased too. The method can be used in situations with multiple treated units or periods, assuming that the underlying factor model is either stationary or cointegrated.

2. The Spatial Synthetic Difference-in-Differences foundation

Synthetic Difference-in-Differences. Arkhangelsky et al. (2021) show that their estimator may be written as the following optimization problem:

$$\left(\widehat{\tau^{sdid}}, \hat{\mu}, \hat{\alpha}, \hat{\beta}\right) = \arg\min_{\mu,\alpha,\beta,\tau} \left\{ \sum_{i=1}^{N} \sum_{t=1}^{T} \left[Y_{it} - (\mu + \alpha_i + \beta_t + \tau D_{it}) \right]^2 \hat{\omega}_i^{sdid} \hat{\lambda}_t^{sdid} \right\}, \quad (1)$$

where we have a two-way fixed effects regression multiplied by nonuniform unit weights $\hat{\omega}_i^{sdid}$ and time weights $\hat{\lambda}_t^{sdid}$. The former comes from the synthetic control algorithm, which is altered compared to the conventional SCM of Abadie and Gardeazabal (2003) and Abadie et al. (2010). On top of that, Arkhangelsky et al. (2021) also introduce time weighting in order to match the average post-treatment outcome with the pre-treatment outcomes for each control unit. The rest of the notation follows Arkhangelsky et al. (2021) with one exception. The treatment status is denoted as D because W used in Arkhangelsky et al. (2021) will be later used for the spatial weights' matrix.

We do not change anything in the way of obtaining $\hat{\omega}_i^{sdid}$ and $\hat{\lambda}_t^{sdid}$. The algorithms to get both types of weights fully follow Arkhangelsky et al. (2021). Hence, in the case of $\hat{\omega}_i^{sdid}$ we have:

$$(\hat{\omega}_0, \hat{\omega}_i^{sdid}) = \arg\min_{\omega_0 \in \mathbb{R}, \omega \in \Omega} \ell_{unit}(\omega_0, \omega), \quad (2)$$

where

$$\ell_{unit}(\omega_0, \omega) = \sum_{t=1}^{T_{pre}} \left(\omega_0 + \sum_{i=1}^{N_{co}} \omega_i Y_{it} - \frac{1}{N_{tr}} \sum_{i=N_{co}+1}^{N} Y_{it} \right)^2 + \zeta^2 T_{pre} ||\omega||_2^2, \quad (3)$$

$$\Omega = \left\{ \omega \in \mathbb{R}_+^N : \sum_{i=1}^{N_{co}} \omega_i = 1, \quad \omega_i = N_{tr}^{-1} \ \forall i = N_{co}+1, ..., N \right\}, \quad (4)$$

with $\mathbb{R}_+^N$ denoting the positive real line while the regularization parameter is set in accordance with Arkhangelsky et al. (2021). The problem described in 2, 3, and 4 deviates from the conventional SCM algorithm from Abadie and Gardeazabal (2003), Abadie et al. (2010) or Abadie et al. (2015) in two aspects. Firstly, including the intercept ω_0 means that we want to fit trends instead of levels. Second, the regularization parameter makes the synthetic unit less sparse; i.e., the weights are more dispersed and hence the synthetic control does not rely on large weights of only few units. What is more, the penalization also helps to ensure the uniqueness of the weights.

The isomorphic minimization problem is set up to find the time weights. The only difference is the absence of the regularization parameter. Thus, the problem follows:

$$(\hat{\lambda}_0, \hat{\lambda}_i^{sdid}) = \arg\min_{\lambda_0 \in \mathbb{R}, \lambda \in \Lambda} \ell_{time}(\lambda_0, \lambda), \tag{5}$$

where

$$\ell_{unit}(\lambda_0, \lambda) = \sum_{i=1}^{N_{co}} \left(\lambda_0 + \sum_{t=1}^{T_{pre}} \lambda_t Y_{it} - \frac{1}{T_{post}} \sum_{t=T_{pre}+1}^{T} Y_{it} \right)^2, \tag{6}$$

$$\Lambda = \left\{ \lambda \in \mathbb{R}_+^T : \sum_{t=1}^{T_{pre}} \lambda_t = 1, \quad \lambda_t = T_{post}^{-1} \ \forall t = T_{pre}+1, ..., T \right\}, \tag{7}$$

while the idea behind using the time weights is to match the average post-treatment outcomes for control units (up to a constant) by reweighting pre-treatment periods. Therefore, only a subset of pre-treatment periods is taken into account.

Spatial Synthetic Difference-in-Differences. In order to extend SyDiD by a spatial component of the treatment spillover, we take a step back and start from the two-way fixed effects estimator, which can be expressed as:

$$\left(\hat{\tau^{did}}, \hat{\mu}, \hat{\alpha}, \hat{\beta} \right) = \arg\min_{\mu, \alpha, \beta, \tau} \left\{ \sum_{i=1}^{N} \sum_{t=1}^{T} \left[Y_{it} - (\mu + \alpha_i + \beta_t + \tau D_{it}) \right]^2 \right\} \tag{8}$$

One can add the spatial extension following Delgado and Florax (2015). Thus, we have:

$$\left(\hat{\tau^{spatialdid}}, \hat{\mu}, \hat{\alpha}, \hat{\beta}, \hat{\tau}, \hat{\rho} \right) = \arg\min_{\mu, \alpha, \beta, \tau, \rho} \left\{ \sum_{i=1}^{N} \sum_{t=1}^{T} \left[Y_{it} - \left(\mu + \alpha_i + \beta_t + \tau(I + \rho W) D_{it} \right) \right]^2 \right\}, \tag{9}$$

where W is a $TN \times TN$ block-diagonal row-standardized spatial weights' matrix that contains non-zero elements for spatial units within a given neighborhood criterion while ρ stands for a spatial autoregressive parameter driving the strength of the spatial interaction in treatment.[4] In other words, equation 9 is the regression from Delgado and Florax (2015) reshuffled as an optimization ordinary least square problem.

[4]Naturally, T stands for the total number of time periods while N denotes the number of units.

One can write $\tau^s = \rho\tau$ and reformulate the expression as follows:

$$\left(\hat{\tau}, \hat{\mu}, \hat{\alpha}, \hat{\beta}, \hat{\tau}_s\right) = \arg\min_{\mu,\alpha,\beta,\tau,\tau_s} \sum_{i=1}^{N} \sum_{t=1}^{T} \left[Y_{it} - \left(\mu + \alpha_i + \beta_t + \tau D_{it} + W\tau^s D_{it}\right) \right]^2, \quad (10)$$

where both τ and τ_s must be estimated. While the former measures a direct treatment effect, the latter an indirect effect. Naturally, if we merge 1 and 10, we yield:

$$\left(\hat{\tau}, \hat{\mu}, \hat{\alpha}, \hat{\beta}, \hat{\tau}_s\right) = \arg\min_{\mu,\alpha,\beta,\tau,\tau_s} \left\{ \sum_{i=1}^{N} \sum_{t=1}^{T} \left[Y_{it} - \left(\mu + \alpha_i + \beta_t + \tau D_{it} + W\tau^s D_{it}\right) \right]^2 \hat{\omega}_i \hat{\lambda}_t \right\} \quad (11)$$

Identification strategy. Equation 11 estimates the parameters τ and τ^s, which we interpret as the average direct and indirect treatment effects, respectively. The interpretation of the parameters relies on a set of assumptions derived from both the Synthetic Difference-in-Differences (SyDiD) and the Spatial Difference-in-Differences (SpDiD) frameworks, along with one specific to our proposed estimator. First, we adopt assumptions that are common to both SyDiD and SpDiD:

A1. *No anticipation:* Units do not adjust their outcomes in advance of the treatment being implemented.

A2. *Parallel trends:* In the absence of treatment, the treated, spillover, and control units would have followed similar trends, conditional on unit and time fixed effects.

Second, we rely on assumptions specific to the spatial DiD literature:

A3. *Additivity and linearity of spillovers:* The potential outcome of a unit depends linearly and additively on its own treatment status and the treatment exposure of its neighbors, captured by the spatial lag $WD_{it} = \sum_j w_{ij} D_{jt}$.

A4. *Limited interference:* Spillover effects occur exclusively through the structure defined by a known, exogenous, spatial structur. No other local or global interference mechanisms are assumed.

Third, we introduce a key assumption that is specific to our estimator:

A5. *Synthetic control transferability:* The synthetic control unit, constructed to approximate the counterfactual outcomes of directly treated units using weights $\hat{\omega}_j$ over the set of pure controls $\mathcal{C}$, can also be used as a valid comparison group for indirectly treated units. That is, for units exposed only to spillovers (i.e., $D_{it} = 0$, $WD_{it} > 0$), we assume that their average untreated potential outcome can be approximated by the same synthetic unit:

$$\frac{1}{|\mathcal{I}_{sp}|} \sum_{i \in \mathcal{I}_{sp}} \mathbb{E}\left[Y_{it}(0, WD_{it})\right] \approx \sum_{j \in \mathcal{C}} \hat{\omega}_j Y_{jt}(0, 0),$$

where $\mathcal{I}_{sp}$ denotes the set of indirectly treated units and $\mathcal{C}$ the set of pure controls. This assumption reflects the idea that units exposed to spillovers tend to be spatially or structurally similar to directly treated units. Therefore, the synthetic unit built to match the untreated potential outcomes of the latter can serve as a reasonable baseline for isolating the spillover effect on the former. Our simulation exercises (Section 3) provide empirical support for this assumption, as the use of shared synthetic weights across directly and indirectly treated units leads to unbiased estimation of both ATT and AITE across different designs.

Estimation. To provide a concise step-by-step summary of what the SpSyDiD estimator is about and how it is generated, we write down a detailed description of all necessary steps. The estimates in equation 11 can be obtained by obeying the following algorithm:

Algorithm 1 Spatial Synthetic Difference-in-Differences

Input: Data for Y and D. Consider N units in total of which N_{tr} are treated directly and N_{sp} indirectly.

Output: $\hat{\tau}$ and $\hat{\tau}^s$

1: Construct the spatial weight matrix W.
2: Separate the donor pool into units receiving spillover (N_{sp}) of treatment and those not receiving any spillover (N_{co}). Let us call the units in the latter group pure controls.
3: Compute regularization parameter ζ following Arkhangelsky et al. (2021).
4: Obtain $\hat{\omega}_i$ and $\hat{\lambda}_t$ stemming from the approach of Arkhangelsky et al. (2021) using only pure control units in the donor pool (N_{co}).
5: For indirectly treated units, use $\hat{\omega}_i = N_{sp}^{-1}$. Substitute $\hat{\omega}_i = N_{tr}^{-1}$ in the case of directly treated units.
6: Compute the SpSyDiD estimator by running the following weighted regression:

$$\left(\hat{\tau}, \hat{\mu}, \hat{\alpha}, \hat{\beta}, \hat{\tau}, \hat{\tau}_s\right) = \arg\min_{\mu,\alpha,\beta,\tau,\tau_s} \left\{ \sum_{i=1}^{N} \sum_{t=1}^{T} \left[Y_{it} - \left(\mu + \alpha_i + \beta_t + \tau D_{it} + W\tau^s D_{it}\right)\right]^2 \hat{\omega}_i \hat{\lambda}_t \right\}$$

The conditional ATE including also the indirect effects. Note that the presence of the treatment spillover alters the conditional ATE definition. If the research question goes beyond the ATT, this happens to be relevant issue.[5] To simplify the notation, consider $t = 1$ being the post-treatment period and $t = 0$ the pre-treatment period.[6] Given that now all units (including directly treated) can be indirectly treated through spillovers coming from the treated units ($0 < wd < 1$ and $wd \in WD$), we have:

$$\begin{aligned} ATE(wd) = &\{\mathbb{E}[Y|D = 1, t = 1, WD = wd] - \mathbb{E}[Y|D = 1, t = 0, WD = wd]\} \\ &-\{\mathbb{E}[Y|D = 0, t = 1, WD = 0] - \mathbb{E}[Y|D = 0, t = 0, WD = 0]\}, \end{aligned} \tag{12}$$

which gives us:

$$ATE(wd) = \tau + \tau^s wd = \tau(1 + \rho wd), \tag{13}$$

[5]The standard Difference-in-Differences (DiD) estimator, as well as the Synthetic Difference-in-Differences (SyDiD) approach of Arkhangelsky et al. (2021), target the Average Treatment Effect on the Treated (ATT) as their primary parameter of interest and do not aim to identify the Average Treatment Effect (ATE). In contrast, our estimator explicitly focuses on the ATE, as it allows us to disentangle and estimate both the direct (ATT) and indirect (spillover) treatment effects. The following paragraphs are dedicated to outlining our approach to identifying and estimating the ATE in this context.

[6]Hence, think in terms of the simplest two-periods textbook DiD case.

where following Delgado and Florax (2015) and working with $\overline{WD}$ as the average proportion of treated neighbors enables us to rewrite it as follows:

$$ATE = \mathbb{E}[ATE(wd)|WD] = \tau(1 + \rho\overline{WD}) \tag{14}$$

Expression 14 shows that the ATE will be biased in case we omit the spillover effects of the treatment. Moreover, we can disentangle the ATE into Average Treatment on the Treated (ATT) and Average Indirect Treatment Effect (AITE). In the case of the former, we get:

$$\begin{aligned} ATT = &\{\mathbb{E}[Y|D = 1, t = 1, WD = 0] - \mathbb{E}[Y|D = 1, t = 0, WD = 0]\} \\ &-\{\mathbb{E}[Y|D = 0, t = 1, WD = 0] - \mathbb{E}[Y|D = 0, t = 0, WD = 0]\}, \end{aligned} \tag{15}$$

which is equivalent to the standard ATE in the case when the SUTVA holds. The AITE can be written as:

$$\begin{aligned} AITE(wd) = &\{\mathbb{E}[Y|D = 0, t = 1, WD = wd] - \mathbb{E}[Y|D = 0, t = 0, WD = wd]\} \\ &-\{\mathbb{E}[Y|D = 0, t = 1, WD = 0] - \mathbb{E}[Y|D = 0, t = 0, WD = 0]\} \end{aligned} \tag{16}$$

Using the notation from 11, we can write $ATT = \tau$ and $AITE(wd) = \tau^s wd$. The later equation can be again reshuffled into $AITE = \tau^s\overline{WD}$. Thus, we can estimate both direct and indirect treatment effects using SpSyDiD.

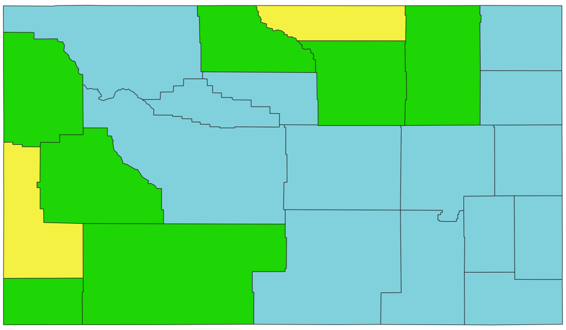
3. Controlled simulations

Outline of the simulations. The goal of this section is to demonstrate that our estimator is suitable for situations of the SUTVA violation. Specifically, we consider the case where treatment affects some of the units included in the control group. If we ignore the treatment spillovers, we end up with a biased and inconsistent estimator of the ATE. In case the estimand of interest is only the Average Treatment Effect on the Treated (ATT), we can exclude the indirectly affected units, as commonly recommended in the literature, and apply SyDiD. However, neither the AITE nor the ATE can be estimated in such a way. The approach of Delgado and Florax (2015) enables doing this, but we show that our estimator can be more efficient as we exploit some of the advantages of SyDiD over the traditional DiD. The idea of Delgado and Florax (2015) is to take the spatial component of treatment into account within the conventional DiD setting. In our approach, we offer an extension for the case of SyDiD in which we reweight the units in the control group by their proximity to the directly treated units. We depict that our estimator can recover the ATT even after taking into account the possibility of the AITE, which is not considered in the estimation process of Arkhangelsky et al. (2021). Moreover, our SpSyDiD can estimate an unbiased AITE, delivering the key value-added feature of the estimator.

3.1 DiD-like simulations at the county level

The case of more treated units. In the first round of simulations, we use the Local Area Unemployment Statistics (LAUS) data from the U.S. Bureau of Labor Statistics at the county level. We consider multiple treated units to resemble a setting closer to DiD studies. We utilize four U.S. states. The number of counties in each state is noted inside the parenthesis in Table 1.

Figure 1: An example of a map depicting direct and indirect treatments



Note: A random iteration of treatment assignment in Wyoming.

We use series for monthly unemployment rate in the period of 2002 to 2004; i.e., we have 36 observations for each county. The treatment occurs at the end of the second year, so we have 24 pre-treatment periods and 12 post-treatment periods. In each iteration, a fixed number of counties (around 10% of the total counties) is treated, and the effect spills over to the immediate neighbors (first-order queen contiguity matrix) of the treated counties. [7] The treated counties are randomly selected in each replica-

[7]To be explicit, we have 2 treated units for Wyoming, 4 for Oregon and 7 for Pennsylvania and

tion, mimicking prior simulation settings in Arkhangelsky et al. (2021) and Delgado and Florax (2015). Figure 1 shows a map of a single iteration for the state of Wyoming. In Figure 1, counties in yellow are treated units, while counties in green receive the spillover effect.

Artificial treatment and spillover effects. In each iteration, we take the real data and adjust the post-treatment period by adding the ATT and the corresponding spillover effect on top of the original series. The magnitude of the ATT is chosen to be 25% of the average unemployment rate of the entire series for a given treated unit. [8] Formally, the observed post-treatment outcome for unit i at time t is defined as:

$$Y_{it}^{(1)} = Y_{it}^{(0)} + \tau \cdot D_{it} + \rho \cdot \tau \cdot (WD_t)_i$$

where:

- $Y_{it}^{(0)}$ is the original unemployment rate in the absence of treatment, drawn from real data;
- $Y_{it}^{(1)}$ is the constructed potential outcome under treatment and spillovers;
- D_{it} is a binary indicator equal to 1 if unit i is directly treated in period t, and 0 otherwise;
- W is the row-standardized first-order queen contiguity matrix;
- $(WD_t)_i$ denotes the proportion of treated neighbors of unit i at time t;
- τ is the direct treatment effect, equal to 25% of the average unemployment rate of unit i in the pre-treatment period;
- ρ is the spillover intensity parameter, set to 0.5.

In pre-treatment periods ($t \leq 24$), outcomes remain unaltered, i.e., $Y_{it}^{(1)} = Y_{it}^{(0)}$. We point out that although we are imposing a simulated treatment effect, the estimation process to obtain the unit and time weights in line with SyDiD is based on the real data. This ensures that the matching structure used remains fully data-driven and not influenced by the simulated effect.

Alabama.

[8] Specifically, this translates to the treatment effect of -1.04 percentage points in the case of Wyoming, -2.04 pps for Oregon, -1.48 pps for Pennsylvania, and -1.72 pps for Alabama.

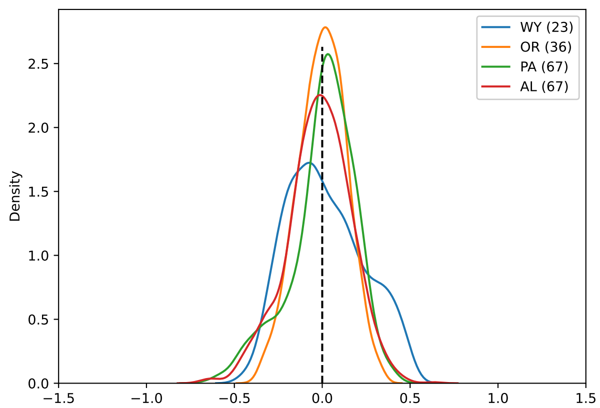
Average Treatment Effect on the Treated. Table 1 shows the results for the estimation of the ATT.[9] We see that the inclusion of the indirectly affected units do not deteriorate the estimation of the ATT. The estimated effects exhibit negligible bias (consistently below 2%) regardless of the sample size. Figure 2 shows the distribution of the errors. Given that we observe Gaussian errors, we preserve the properties of the original SyDiD estimator.

Table 1: The ATT bias using Spatial Synthetic DiD (relative)

Wyoming (23)	**Oregon (36)**	**Pennsylvania (67)**	**Alabama (67)**
0.018	0.001	0.003	-0.019

Source and note: The Local Area Unemployment Statistics (LAUS) data from the U.S. Bureau of Labor Statistics. The results of the ATT bias in relative terms using SpSyDiD.

Figure 2: ATT - Distribution of errors



Source and note: The Local Area Unemployment Statistics (LAUS) data from the U.S. Bureau of Labor Statistics. Errors distribution for simulations of the ATT for all states.

Average Indirect Treatment Effect. Besides the ATT, we also estimate the AITE. What is more, we argue that the ability to obtain the AITE is the main advantage of our estimator. We consider the spillover effects to be additive. Specifically, treated units may receive the indirect effect if they have other treated neighbors. The results are shown in Table 2.

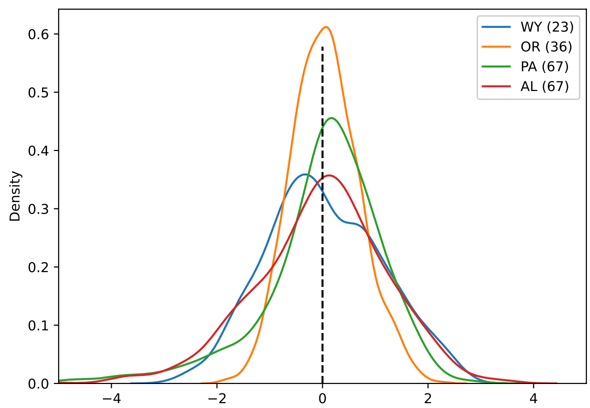
[9] We run 1000 simulations for each state.

Table 2: The AITE bias using Spatial Synthetic DiD (relative)

Wyoming (23)	**Oregon (36)**	**Pennsylvania (67)**	**Alabama (67)**
0.026	0.055	0.013	-0.038

Source and note: The Local Area Unemployment Statistics (LAUS) data from the U.S. Bureau of Labor Statistics. The results of the AITE bias in relative terms using SpSyDiD.

Figure 3: AITE - Distribution of errors



Source and note: The Local Area Unemployment Statistics (LAUS) data from the U.S. Bureau of Labor Statistics. Errors distribution for simulations of the AITE for all states.

The indirect treatment effect due to the presence of spillovers is estimated with a high accuracy for all four states. The relative bias is in the range of 1.3-5.5%. Note that we intentionally choose a wide range of the number of counties in each state to consider different DiD-like setups. Figure 3 shows the distribution of errors for the case of the AITE. We see that the plot is similar to the ATT figure. In the case of AITE, as we aim to proceed with inference, we run a normality test for the errors distribution. Normality is rejected for Wyoming, Alabama and Pennsylvania, and not rejected for Oregon. This can be attributed to the simulation design that did not impose any constraint on the data, apart from including the effect. In other words, we are simulating scenarios in which normality of errors is typically assumed but cannot be tested[10]. The result show that under certain conditions, we have Gaussian errors. To conclude, our estimator can accurately estimate both the ATT and the AITE and can be used for valid inference.

[10] For the sake of comparison, in Arkhangelsky et al. (2021) data is generated with noise that is both Gaussian and homoskedastic across units

3.2 SCM-like simulations at the state level

The case of one treated unit. The previous subsection shows that our estimator can recover the estimands of interests (ATT and AITE) if we have many treated units. What is next, we consider the SCM-like example with only one treated unit. In this round of simulations, we use data at the state level.

Artificial treatment and spillover effects. We again work with the same dataset of the Local Area Unemployment Statistics (LAUS) from the U.S. Bureau of Labor Statistics. Hence, we have the unemployment rate data with 24 pre-treatment periods and 12 post-treatment periods. In this case, we do not draw a random state to be treated. Instead, we iterate across all possible units and consider one state treated every time. We also vary the time period of the treatment and, thus, the whole period that we work with. Specifically, we iterate the total period of 36 months across the window from the beginning of 1976 to the end of 2015.

Consequentially, we have 49 states (Hawaii excluded) and 40 years, leading to 1960 combinations of the estimation. We estimate the ATT utilizing all three methods - SyDiD, Spatial DiD, and SpSyDiD. Naturally, the AITE is estimated only by using Spatial DiD of Delgado and Florax (2015) and our SpSyDiD.

Average Treatment Effect on the Treated. Firstly, Table 3 depicts the results of the estimation of the ATT, and Table 4 presents the Root Mean Squared Error (RMSE) for each estimator. In the previous DiD-like simulations, the ATT estimated by Synthetic DiD cannot be exactly compared to our approach, given that the treated units that also receive spillover must be removed from the analysis. However, now running a perfectly comparable scenario (just one treated unit at a time), we see that the estimation of the ATT is very similar for both methods. What is more, the estimation is superior to Spatial DiD of Delgado and Florax (2015) in terms of the efficiency. As before, Figure 4 shows density plots with the distributions of the bias for all three estimators.

Table 3: Descriptive statistics of the bias of the ATT (relative)

Statistic	SyDiD	SpSyDiD	SpDiD
count	1960	1960	1960
mean	0.004	0.004	< 0.001
std	0.195	0.195	0.384
min	-0.939	-0.938	-2.165
25%	-0.114	-0.114	-0.216
50%	-0.001	0	0.012
75%	0.118	0.118	0.226
max	1.023	1.019	1.751

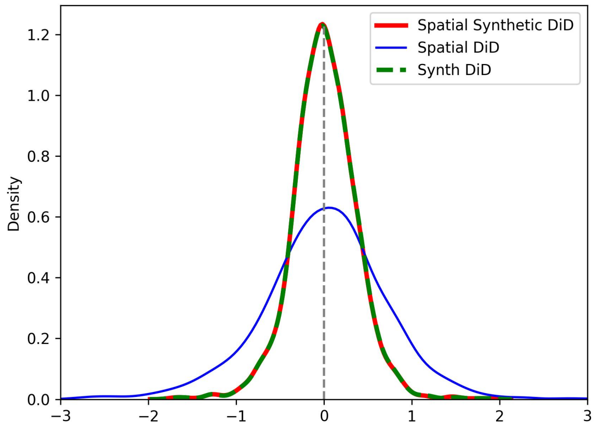
Source and note: The Local Area Unemployment Statistics (LAUS) data from the U.S. Bureau of Labor Statistics. Descriptive statistics of the ATT bias in relative terms.

Table 4: RMSE of the ATT estimation

Estimator	SyDiD	SpSyDiD	SpDiD
RMSE	0.361	0.362	0.712

Note: RMSE of the ATT bias in level terms.

Figure 4: Density plot of the bias for the ATT



Note: The Local Area Unemployment Statistics (LAUS) data from the U.S. Bureau of Labor Statistics. Density plot of the bias for the ATT comparing all three methods - Synthetic Difference-in-Differences, Spatial Difference-in-Differences, and Spatial Synthetic Difference-in-Differences.

Average Indirect Treatment Effect. It is clear that Spatial DiD is much less precise than the other two estimators. The results are similar when analyzing the estimation of the AITE. Table 5 shows the simulation's descriptive statistics for the bias, Table 6 compares the RMSE, and Figure 5 again demonstrates the density plot.

Table 5: Descriptive statistics of the bias of the AITE (relative)

Statistic	SpSyDiD	SpDiD
count	1960.0	1960.0
mean	0.01	-0.0198
std	0.7379	1.4864
min	-3.4611	-7.8947
25%	-0.4107	-0.7536
50%	0.0059	0.0236
75%	0.4148	0.7975
max	3.2999	5.6462

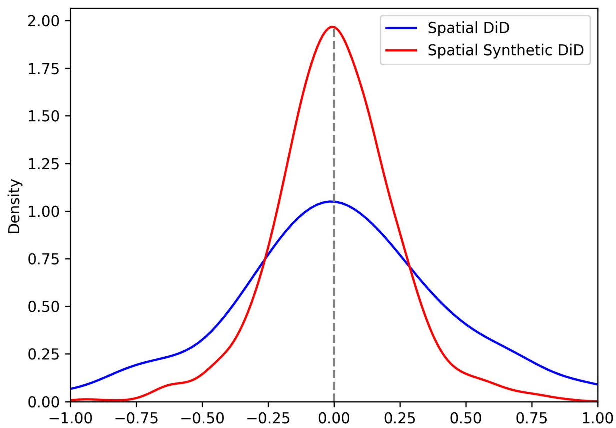
Source and note: The Local Area Unemployment Statistics (LAUS) data from the U.S. Bureau of Labor Statistics. Descriptive statistics of the AITE bias in relative terms.

Table 6: RMSE of the AITE estimation

Estimator	SpSyDiD	SpDiD
RMSE	0.231	0.473

Note: RMSE of the AITE bias in level terms.

Figure 5: Density plot of the bias for the AITE



Note: The Local Area Unemployment Statistics (LAUS) data from the U.S. Bureau of Labor Statistics.. Density plot of the bias for the AITE comparing two spatial extensions of DiD - Spatial Difference-in-Differences and Spatial Synthetic Difference-in-Differences.

Our SpSyDiD reveals to be more efficient than Spatial DiD. When comparing our estimator to Delgado and Florax (2015), it is straightforward to see their approach as a special case of ours. Particularly, using uniform weights for units and time periods means that 11 boils down back to 10. It is thus natural to conclude that all the features presented in Arkhangelsky et al. (2021) related to the comparison of Synthetic Difference-in-Differences and conventional Difference-in-Differences carry forward when analyzing the direct treatment effect by our estimator or Delgado and Florax (2015).

Observing the results of the AITE, it is evident that even though we use the synthetic series created for the directly treated unit as the control unit also for the indirect treated units, the results are superior to the uniform weighting of the control set. This fact supports our assumption that closer units, i.e., the ones that receive the spillover effect, are similar enough to the directly treated unit to be compared to the same synthetic series.

4. A real world example

Introduction. In what follows, we replicate Bohn et al. (2014) and investigate the possibility of spillover effects from the treatment by utilizing our Spatial Synthetic Difference-in-Differences estimator. Bohn et al. (2014) estimate the effect of the 2007 Legal Arizona Workers Act (LAWA) on the share of the state's population characterized as noncitizen Hispanic.[11] They find notable declines in the proportion of the population after the intervention.

Regarding the spillover effects, the authors acknowledge the possibility of these movements potentially biasing the results, but they try to use alternative identification strategies to circumvent the problem. Additionally, they claim that even if any spillover effects exist, they might be quantitatively unimportant.

We want to demonstrate that these spillover effects, what we call the AITE, can be of non-negligible magnitude. We offer a way to explicitly take them into account by estimating them without involving any harm in the estimation of the direct treatment

[11] The Legal Arizona Workers Act prohibits intentionally employing unauthorized noncitizen workers. It also requires employers to verify employment authorization of each new employee through a federal database.

effect.[12]

The original results. Bohn et al. (2014) utilize the Synthetic Control Method of Abadie and Gardeazabal (2003) and Abadie et al. (2010) to estimate the ATT. The authors provide the resulting effects on several subgroups of the population, with the varying size of the effect across them. We focus our analysis on the variable that represents the highest effect in Bohn et al. (2014) - the proportion of noncitizen Hispanic aged between 15 and 45 years old. In Table 3 of their paper, Bohn et al. (2014) document the decrease of 2.6 percentage points in the proportion of the subpopulation of interest. In the 2 years before the intervention, the proportion of noncitizen Hispanic aged 15 to 45 in Arizona was about 14%; i.e, the effect was the reduction of almost 18% in relative terms. We hypothesize that the population that left Arizona might have moved to neighboring states, which would generate a spillover effect of the opposite sign.

SCM without a large set of predictors. Bohn et al. (2014) deploy the Synthetic Control Method in its standard way described in Abadie and Gardeazabal (2003), Abadie et al. (2010), and Abadie et al. (2015). Hence, the authors fit the synthetic series using the dependent variable and several other predictors.[13] In our method, we fit the synthetic series using solely the dependent variable, following Arkhangelsky et al. (2021). In what is next, we demonstrate that there is no significant loss when we do not use the predictors from Bohn et al. (2014) apart from the variable of interest. Table 7 compares the results.

Although in our real world example we focus only on one variable, we estimate the effect using SCM without predictors for all variables of interest from Bohn et al. (2014). We see only cosmetic changes in the two scenarios. Thus, we can conclude that it is rather a safe approach to proceed in line of Arkhangelsky et al. (2021) and to avoid other predictors in constructing the synthetic control unit.

[12] Bear in mind that in this particular setup, the ATE is not an additive combination of the ATT and AITE. However, for most of cases where the ATE can be composed in such a way, our estimator offers a way to estimate the overall ATE.

[13] They specifically work with the average values of the proportion of the state workforce in each of 9 industrial categories, the proportion of the state population in each of four broad educational attainment categories (less than high school, high school graduate, some college, college or more), and the state unemployment rate.

Table 7: Comparison of SCM with and without predictors

Variable of interest	Original (pps)	No predictors (pps)
% of noncitizen Hispanic, aged 15 to 45	-2.65	-2.66
% of noncitizen Hisp., 15-45, high school or less	-2.04	-2.09
% of noncitizen Hispanic, over 15	-1.34	-1.51
% of noncitizen Hisp., over 15, high school or less	-1.18	-1.23
% of overall noncitizen Hispanic	-1.48	-1.56

Source and note: Based on the databases of the Current Population Survey (CPS) between 1998 and 2009. Results for all variables of interest using the Synthetic Control Method with and without predictors (in pps).

The difference of the ATT between SyDiD and SpSyDiD. Spatial Synthetic Difference-in-Differences builds on the Synthetic Difference-in-Differences estimator of Arkhangelsky et al. (2021). Therefore, we redo the estimations from Bohn et al. (2014) using the Synthetic Difference-in-Differences of Arkhangelsky et al. (2021) and compare it to the Synthetic Control Method of Abadie et al. (2010) (both with and without using the predictors). Table 8 depicts the results.

Table 8: Comparison of SyDiD against SCM with and without predictors

Variable of interest	Original (pps)	No predictors (pps)	SyDiD (pps)
% of noncitizen Hispanic, aged 15 to 45	-2.65	-2.66	-2.90
% of noncitizen Hisp., 15-45, high school or less	-2.04	-2.09	-2.57
% of noncitizen Hispanic, over 15	-1.34	-1.51	-1.95
% of noncitizen Hisp., over 15, high school or less	-1.18	-1.23	-1.68
% of overall noncitizen Hispanic	-1.48	-1.56	-1.84

Source and note: Based on the databases of the Current Population Survey (CPS) between 1998 and 2009. Results for all variables of interest using the Synthetic Control Method with and without predictors and Synthetic Difference-in-Differences (in pps).

Interestingly, the SyDiD estimator generates stronger effects than SCM for all variables of interest. The effects under SyDiD are roughly 40 basis points higher. We attribute this to the fact that in SCM, California receives the highest weight for the case with predictors while Texas for the situation without predictors (with California to be the second most important). What is more, both California and Texas show a drop of the proportion of noncitizen Hispanic subpopulation around the time of the LAWA. While the decreasing trend at least reverts subsequently for Texas, the proportion continues to decline in the case of California. Consequentially, this may possibly bias the ATT downward. SyDiD is less prone to such a bias stemming from reliance on one or two control unit substantially, given it features a greater weights dispersion.

The possibility of spillover effects. We consider to be possible that part of the subpopulation of interest moved from Arizona to neighboring states. In case of any suspicion that this might happen, the standard way to handle the problem is to exclude the potentially indirectly treated units from the set of control units. Bohn et al. (2014) do so in their article, yet not for all variables of interest. Based on such a robustness check for some of the variables of interest, they claim the insignificance of possible spillover effects. Table 9 compares the results when we keep neighboring states in the donor pool and when we exclude them for the SyDiD estimator. The results are very similar, which is also the case for Bohn et al. (2014) even if they use SCM.

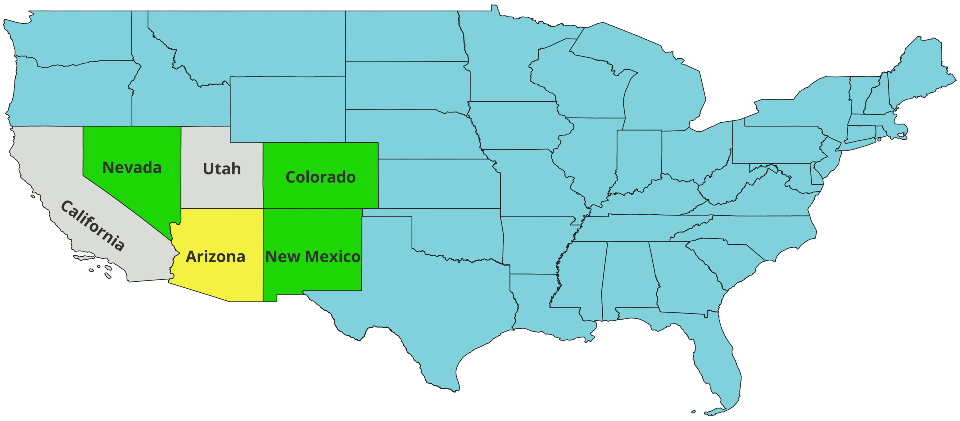
Table 9: Comparison of SyDiD with complete donor pool against removing neighbors

Variable	SyDiD (pps)	SyDiD no neighbors (pps)
% of noncitizen Hispanic, aged 15 to 45	-2.90	-2.739
% of noncitizen Hisp., 15-45, high school or less	-2.57	-2.504
% of noncitizen Hispanic, over 15	-1.95	-1.975
% of noncitizen Hisp., over 15, high school or less	-1.68	-1.727
% of overall noncitizen Hispanic	-1.84	-1.836

Source and note: Based on the databases of the Current Population Survey (CPS) between 1998 and 2009. Results for all variables of interest using Synthetic Difference-in-Differences with having neighbors of Arizona in the donor pool and with excluding them (in pps).

We argue that similar results for the ATT when removing neighboring states are not enough to rule out the existence of spillover effects, if they are the matter of interest for us. In the case of our dataset, the states that might receive spillover effects do not get any relevant weight in the SCM algorithm; i.e., the ATT can be unbiasedly recovered but the AITE can still an additional insightful information about the movement of noncitizen Hispanic subpopulation across other states after the LAWA implementation.

Figure 6: Directly and indirectly treated states



Note: United States Map highlighting the treated state of Arizona and all its neighbors. The states that we consider in our spillover estimation are the green ones.

To focus on our sample, we can see that the state of Arizona has 5 bordering states: California, Nevada, Utah, Colorado, and New Mexico. Figure 6 shows the map of the USA with the incriminated states highlighted. Utah was excluded from the sample because it implemented a similar law. Due to this fact, it should be unlikely that people move from Arizona to Utah. Our hypothesis is that people moved to the remaining 4 states, while we want to estimate this possible spillover effect. Bohn et al. (2014) clearly demonstrate that because of the population size of California, even if all migrants from Arizona moved there, the result in the proportion of noncitizen Hispanic would be imperceptible, and almost impossible to be identifiable by the data. Therefore, we exclude California from our analysis and focus our attention on estimating the effect on the three other states, Nevada, Colorado, and New Mexico. Note that if we find some effect, it may be considered as a lower bound, given that it is highly probable that other people would also move to California, the state with the second-highest proportion of noncitizen Hispanic in the whole country.

The Spatial Synthetic Difference-in-Differences. We are interested in the estimation of the spillover effect of one specific variable of interest - the proportion of noncitizen Hispanic at working age (15-45). Besides the fact that it is the variable with the strongest effect, and therefore has the biggest probability to generate an identifiable spillover effect (considered as a scale of the main effect), we have theoretical reasons to believe this specific group might be more prone to rapid move to another state. The Legal Arizona Workers Act is an employer-targeted law, which requires employers to verify the identity and work eligibility of all new hires. Hence, it is reasonable to assume that the LAWA should affect primarily people at working age.

Although we are primarily investigating the possibility of the spillover effect on one variable, we run the Spatial Synthetic Difference-in-Differences for all variables studied in Bohn et al. (2014) for the sake of completeness. Table 10 shows the ATT estimated by the Synthetic Difference-in-Differences when removing neighbors (our benchmark) and both the ATT and AITE estimated by the Spatial Synthetic Difference-in-Differences.

Table 10: The ATT using SyDiD and SpSyDiD; the AITE using SpSyDiD

Variable	ATT-SyDiD	ATT-SpSyDiD	AITE-SpSyDiD
% of noncitizen Hispanic, aged 15 to 45	-2.739	-2.736	0.857
% of noncitizen Hisp., 15-45, high school or less	-2.504	-2.502	0.501
% of noncitizen Hispanic, over 15	-1.975	-1.973	0.209
% of noncitizen Hisp., over 15, high school or less	-1.727	-1.725	0.152
% of overall noncitizen Hispanic	-1.836	-1.834	0.404

Source and note: Based on the databases of the Current Population Survey (CPS) between 1998 and 2009. Results for all variables of interest using Synthetic Difference-in-Differences (ATT) and Spatial Synthetic Difference-in-Differences (ATT and AITE) (in pps).

Firstly, we see that the values of the ATT are almost the same for both estimators. This confirms our previous statement that the Spatial Synthetic Difference-in-Differences does not significantly harm the estimation of the ATT. In terms of the AITE, the estimator identifies possible indirect effects for all variables. The sign is in line with expectations for each of them. The results suggest that people left Arizona and moved to the neighboring states. The main variable of interest shows the relative spillover effect regarding the direct treatment effect of roughly 30%.

Inference. To conduct inference of our results, we follow Arkhangelsky et al. (2021) when it comes to estimating the ATT. However, we also propose to deploy the placebo-based inference for the AITE. The main idea of the test is to vary the treatment across both exposed and unexposed units. Abadie et al. (2010) and Abadie et al. (2015) propose to compare the ratio of the root-mean-square error (RMSE) in the post and pre-treatment period. The exposed unit should have the highest (or among the highest) value of the ratio. We utilize similar knowledge and in lines with Arkhangelsky et al. (2021) use placebo predictions (of unexposed units) to estimate noise level and hence to obtain the variance estimator.

What is more, we use the placebo variance estimator also for the indirect effect. The inference can be applied in an equivalent way, as we document in the outlined algorithm below. Bear in mind that the indirect effect inference conditions on a given fixed exposed unit(s). Put differently, the directly treated units stay unchanged across the placebo rationing.

Algorithm 2 Placebo Variance Estimation of the AITE

Input: Data for Y_{co} and the number of placebo iterations B. Consider N units in total of which N_{tr} are treated directly and N_{sp} indirectly.

Output: Variance estimator $\hat{V}_{\tau^s}^{\text{placebo}}$

for $b \leftarrow 1$ **to** B **do**:
Sample out N_{tr} and N_{sp} without replacement.
Construct a placebo treatment matrix $\mathbf{D}_{co}^{(b)}$ for the pure controls $(N - N_{sp} - N_{tr})$.
Compute the SpSyDiD estimator $\hat{\tau}^{(s,b)}$ based on $(\mathbf{Y}_{co}, \mathbf{D}_{co}^{(b)})$
end
Define $\hat{V}_{\tau^s}^{\text{placebo}} = \frac{1}{B}\sum_{b=1}^{B}\left(\hat{\tau}^{(s,b)} - \frac{1}{B}\sum_{b=1}^{B}\hat{\tau}^{(s,b)}\right)^2$

We use the place variance estimate for all variables in table 10. We conduct the inference tests for both the ATT and AITE using our SpSyDiD estimator by keeping the number of placebo neighbors to be equal to the number of neighbors of Arizona. Thus, the same number of neighbors is randomly drawn from the set of the pure control units; we run 11000 combinations of the placebo AITE. The standard errors are shown in the brackets under the coefficient values in table 11.

Table 11: Inference in SpSyDiD

Variable	ATT-SpSyDiD	AITE-SpSyDiD
% of noncitizen Hispanic, aged 15 to 45	-2.736	0.857
	(0.795)	(0.477)
% of noncitizen Hispanic, 15-45, high school or less	-2.502	0.501
	(0.677)	(0.436)
% of noncitizen Hispanic, over 15	-1.973	0.209
	(0.542)	(0.286)
% of noncitizen Hispanic, over 15, high school or less	-1.725	0.152
	(0.461)	(0.266)
% of overall noncitizen Hispanic	-1.834	0.404
	(0.498)	(0.271)

Source and note: Based on the databases of the Current Population Survey (CPS) between 1998 and 2009. Results for all variables of interest using Spatial Synthetic Difference-in-Differences (ATT and AITE) (in pps). Standard errors are in the brackets below the coefficient values.

5. Conclusion

We offer an extension of the Synthetic Difference-in-Differences estimator of Arkhangelsky et al. (2021) for situations of treatment spillover. In such a case, the Stable Unit Treatment Value Assumption is violated; hence, the Average Treatment Effect estimation will be biased if we use the standard Synthetic Difference-in-Differences. We exploit the spatial version of Difference-in-Differences of Delgado and Florax (2015) and include their structure that builds on the spatial weights' matrix from the spatial econometrics literature (Anselin, 1988) into Synthetic Difference-in-Differences. Hence, the resulting regression equation may be summarized as the SLX model (Vega and Elhorst, 2015) estimated by a weighted least square given the reweighting of control units and pre-treatment periods coming from Arkhangelsky et al. (2021). Following this strategy, we can disentangle the Average Treatment Effect (ATE) into direct and indirect effects.

Using controlled simulations, we show that our estimator can handle the situation of the SUTVA violation, and get rid of the bias caused by omitted variable problem, by utilizing the spatial treatment spillover. When comparing to Arkhangelsky et al. (2021), we bring the possibility of allowing the option to estimate the additional component of the ATE - the Average Indirect Treatment Effect (AITE). Thus, we get a better approximation of the ATE if there is any treatment spillover. Comparing to Delgado and Florax (2015), we show that exploiting the weighting features of the Synthetic Difference-in-Differences delivers more precision to the estimation of both the Average Treatment Effect on the Treated (ATT) and the AITE.

We deploy our Spatial Synthetic Difference-in-Differences to a real-world example following up on the estimation of the effect of the 2007 Legal Arizona Workers Act on the share of noncitizen Hispanic subpopulation in Bohn et al. (2014). While the authors estimate the ATT, we additionally track certain movements of people leaving Arizona by estimating the AITE on neighboring states. We show that the indirect effects on neighboring states are significant for working subpopulation.

Bibliography

Abadie, A., Diamond, A. and Hainmueller, J. (2010), 'Synthetic control methods for comparative case studies: Estimating the effect of california's tobacco control program', *Journal of the American Statistical Association* **105(490)**, 493–505.

Abadie, A., Diamond, A. and Hainmueller, J. (2015), 'Comparative politics and the synthetic control method', *American Journal of Political Science* **59(2)**, 495–510.

Abadie, A. and Gardeazabal, J. (2003), 'The economic costs of conflict: A case study of the basque country', *American Economic Review* **93(1)**, 113–132.

Abadie, A. and L'Hour, J. (2021), 'A penalized synthetic control estimator for disaggregated data', *Journal of the American Statistical Association* **116(536)**, 1817–1834.

Angrist, J. D., Imbens, G. W. and Rubin, D. B. (1996), 'Identification of causal effects using instrumental variables', *Journal of the American Statistical Association* **91(434)**, 444–455.

Anselin, L. (1988), 'Spatial econometrics: Methods and models', *Kluwer Academic, Dordrecht* .

Arkhangelsky, D., Athey, S., Hirshber, D. A., Imbens, G. W. and Wager, S. (2021), 'Synthetic difference-in-differences', *American Economic Review* **111(12)**, 4088–4118.

Bohn, S., Lofstrom, M. and Raphael, S. (2014), 'Did the 2007 legal arizona workers act reduce the state's unauthorized immigrant population?', *The Review of Economics and Statistics* **96(2)**, 258–269.

Butts, K. (2021), 'Difference-in-differences estimation with spatial spillovers', *Papers 2105.03737, arXiv.org* .

Butts, K. (2023), 'Difference-in-differences with geocoded microdata', *Journal of Urban Economics: Insights* **133**, 103493.

Cao, J. and Dowd, C. (2019), 'Estimation and inference for synthetic control methods with spillover effects', *Papers 1902.07343, arXiv.org* .

Clarke, D. C. (2017), 'Estimating difference-in-differences in the presence of spillovers', *MPRA Paper 81604, University Library of Munich, Germany* .

Delgado, M. S. and Florax, R. J. (2015), 'Difference-in-differences techniques for spatial data: Local autocorrelation and spatial interaction', *Economics Letters* **137**, 123–126.

Di Stefano, R. and Mellace, G. (2024), 'The inclusive synthetic control method', *arXiv:2403.17624* .

Dubé, J., Legros, D., Thériault, M. and Des Rosiers, F. (2014), 'A spatial difference-in-differences estimator to evaluate the effect of change in public mass transit systems on house prices', *Transportation Research Part B: Methodological* **64**, 24–40.

Kolak, M. and Anselin, L. (2020), 'A spatial perspective on the econometrics of program evaluation', *International Regional Science Review* **43(1-2)**, 128–153.

Rubin, D. B. (1990), 'Formal mode of statistical inference for causal effects', *Journal of Statistical Planning and Inference* **25(3)**, 279–292.

Vega, S. and Elhorst, J. P. (2015), 'The slx model', *Journal of Regional Science* **55(3)**, 339–363.

Appendix

Table 12: Weights of states for % of noncitizen Hispanic, aged 15 to 45

State	Weight
North Carolina	19.30%
Missouri	18.00%
New Jersey	15.85%
Maryland	11.11%
Idaho	10.16%
New York	07.74%
Illinois	05.09%
Louisiana	02.94%
Alabama	02.93%
Georgia	02.51%
Nebraska	01.77%
Oregon	01.28%
Delaware	00.76%
Indiana	00.56%

Table 13: Weights of years for % of noncitizen Hispanic, aged 15 to 45

Year	Weight
2007	68.40%
2006	17.80%
2005	13.80%

Source and note: Based on the databases of the Current Population Survey (CPS) between 1998 and 2009. The unit and time weights of synthetic control units for % of noncitizen Hispanic, aged 15 to 45.

Table 14: Weights of states for % of noncitizen Hispanic, 15-45, high school or less

State	Weight
North Carolina	32.91%
Maryland	19.69%
Georgia	14.48%
Idaho	12.65%
Washington	12.03%
Oregon	4.07%
Louisiana	3.61%
Alabama	0.43%
New Jersey	0.13%

Table 15: Weights of years for % of noncitizen Hispanic, 15-45, high school or less

Year	Weight
2007	69.41%
2005	10.79%
2004	10.62%
2000	8.15%
1999	1.04%

Source and note: Based on the databases of the Current Population Survey (CPS) between 1998 and 2009. The unit and time weights of synthetic control units for % of noncitizen Hispanic, 15-45, high school or less.

Table 16: Weights of states for % of noncitizen Hispanic, over 15

State	Weight
North Carolina	16.28%
New Jersey	13.03%
Georgia	11.62%
Missouri	9.78%
Idaho	9.17%
Maryland	8.66%
Alabama	6.33%
Oregon	5.37%
New York	4.67%
Nebraska	4.49%
Washington	4.09%
Louisiana	2.79%
Kentucky	1.82%

Table 17: Weights of years for % of noncitizen Hispanic, over 15

Year	Weight
2007	65.47%
2004	22.22%
2002	10.58%
2003	1.73%

Source and note: Based on the databases of the Current Population Survey (CPS) between 1998 and 2009. The unit and time weights of synthetic control units for % of noncitizen Hispanic, over 15.

Table 18: Weights of states for % of noncitizen Hispanic, over 15, high school or less

State	Weight
North Carolina	21.89%
New Jersey	17.78%
Idaho	13.84%
Georgia	13.69%
Nebraska	12.41%
New York	11.82%
Maryland	3.22%
Oregon	2.72%
Missouri	1.98%
Washington	0.49%
Kentucky	0.17%

Table 19: Weights of years for % of noncitizen Hispanic, over 15, high school or less

Year	Weight
2007	70.80%
2003	14.24%
2005	4.13%
2004	3.88%
2000	3.77%
2002	3.19%

Source and note: Based on the databases of the Current Population Survey (CPS) between 1998 and 2009. The unit and time weights of synthetic control units for % of noncitizen Hispanic, over 15, high school or less.

Table 20: Weights of states for overall % of noncitizen Hispanic

State	Weight
North Carolina	19.30%
Missouri	18.01%
New Jersey	15.85%
Maryland	11.11%
Idaho	10.16%
New York	7.74%
Kentucky	5.09%
Louisiana	2.94%
Alabama	2.92%
Georgia	2.51%
Nebraska	1.77%
Oregon	1.28%
Delaware	0.76%
Kentucky	0.57%

Table 21: Weights of years for overall % of noncitizen Hispanic

Year	Weight
2007	68.44%
2004	17.80%
2002	13.76%

Source and note: Based on the databases of the Current Population Survey (CPS) between 1998 and 2009. The unit and time weights of synthetic control units for overall % of noncitizen Hispanic.