NÁRODNÁ BANKA SLOVENSKA
EUROSYSTÉM

Factor Efficiency in CES Production Function

Constant Elasticity of Substitution production function allows us, compared with a Cobb-Douglas function, to model different efficiency trends of labour and capital, and to model the private and public sectors independently. We also find a function with separate trends based on Box-Cox transformations outperforms capturing technological progress by a single exponential trend. Labour and capital efficiency gains in our preferred model are stable or increasing over time in the private sector, whereas they are gradually decreasing in the public sector. The elasticity of substitution between labour and capital is significantly greater than zero, but also lower than one half.

Introduction

Production functions are a standard tool used for modelling production process on aggregate level. They determine output or value added from production factors – labor and capital, and in addition they include technological progress (or total factor productivity). Cobb-Douglas production function (further as CDF) and CES production function (further as CES) are the most widely used in macroeconomic applications.

CDF production function is the preferred specification in most applications, since it is easy to estimate by ordinary least squares. It leads to constant shares of labor costs and capital costs in output, which is a testable stylized fact. This is because elasticity of substitution between factors (e.g. labor and capital) is assumed to be equal to one.[1]

If shares of labor and capital tend to vary in time, CES production function may be a preferred modelling alternative to CDF. The functional form of the CES production function is non-linear and its estimation is more complex. The efficiency of its estimation can be increased by estimating it in a system with other equation(s), containing parameters of interest. We estimate supply systems for both private and public sector in Slovakia, allowing the elasticity of substitution to take any positive value and exploring two ways of modeling technical progress for each sector.

Choice of supply system

We use the CES production function, since the factor shares in output are not constant.[2] Technical progress for both sectors is captured twofold: i) in a system with a simple

[1] Further details to production function and elasticity of substitution can be found e.g. in Chang (1984).

[2] Variability of labor cost ratio to value added is observable in data.

exponential trend and ii) with a Box-Cox transformation.[3] The equations (1) and (2), resp. (1) and (3) are combined into a single equation in estimations.

$$\left(\frac{Q}{\bar{Q}}\right) = A_t\left[a\breve{K}^{\rho} + (1-a)\breve{L}^{\rho}\right]^{\frac{1}{\rho}}, \qquad \text{where } \rho = \frac{\sigma-1}{\sigma} \qquad (1)$$

For the two versions with exponential trends,

$$A_t = A_0\exp[\psi(t-\bar{t})]\,,\ \breve{K} = \left(\frac{K}{\bar{K}}\right) \text{ and } \breve{L} = \left(\frac{L}{\bar{L}}\right) \qquad (2)$$

For the two versions with Box-Cox transformation,

$$A_t = A_0\,,\ \breve{K} = \left(\frac{K}{\bar{K}}\right) exp\left\{\frac{\bar{t}\gamma_1}{\lambda_1}\left[\left(\frac{t}{\bar{t}}\right)^{\lambda_1} - 1\right]\right\} \text{and } \breve{L} = \left(\frac{L}{\bar{L}}\right) exp\left\{\frac{\bar{t}\gamma_2}{\lambda_2}\left[\left(\frac{t}{\bar{t}}\right)^{\lambda_2} - 1\right]\right\} \qquad (3)$$

The second equation in each system is a dynamized relative demand function. This function relates the ratio of factor volumes to the ratio of factor prices and separate time trends, if they are used in the production function.[4]

$$log\left(\frac{K}{L}\right) = C_0 + \nu log\left(\frac{K_{t-1}}{L_{t-1}}\right) + (1-\nu)\,\sigma log\left(\frac{w}{q}\right) + (1-\nu)(\sigma-1)\left\{\frac{\bar{t}\gamma_1}{\lambda_1}\left[\left(\frac{t}{\bar{t}}\right)^{\lambda_1} - 1\right] - \frac{\bar{t}\gamma_2}{\lambda_2}\left[\left(\frac{t}{\bar{t}}\right)^{\lambda_2} - 1\right]\right\} \quad (4)$$

The CES production function has a constant term A_0, that is by construction bound to be near unity. The parameter a distributes weights between capital and labor (labor weight is denoted as $1\text{-}a$). The parameter σ is the elasticity of substitution, which measures the ease of substitution between capital and labor, and vice versa, and the elasticity of ratio of factor volumes to ratio of factor prices. The parameter ψ is the annual rate of technical progress when it is modeled as an exponential trend. Parameters γ_i and λ_i determine the shape of separate trends for capital (i=1) and labor (i=2).[5] The parameter C_0 is the constant term in the relative demand function and ν is the parameter of autoregressive term in this equation and measures the inertia of the ratio of factor volumes. The base period relates to the date to which indices used in the production function are fixed (see Appendix for details).

Results of estimation

The purpose of using Box-Cox transformations as separate trends for labor and capital is to allow the efficiency trend to have a more general path than the one implied by a single time trend. By merging (2) into (1) together with equation (4) we get the supply system A with simple exponential trend. Respectively, by merging (3) into (1) together with equation (4) we get the supply system B with Box-Cox transformation. Separate estimations for private and public sector were carried out for both versions of the supply system, leading to four sets of estimated parameters, presented in Table 1.

For the private sector, the elasticity of substitution is high, (practically equal to unity) in the system Priv-A, it is much lower for the specification Priv-B. If we assumed that the technical progress is sufficiently captured with an exponential trend, we could use the CDF instead of the CES function.[6]. However, if we introduce efficiency trends[7] for individual production

[3] Technical details of Box-Cox transformation are available e.g. in Klump et al. (2004).

[4] In systems with single exponential trend, the last term is not used, the complete equation is used in systems with Box-Cox transformations.

[5] See Klump et al. (2004) for details.

[6] This finding is consistent with Willman (2002). We positively proved that the CES function with exponential trend can be replaced with the Cobb-Douglas function by estimating it and comparing the distribution of the residuals.

factors, the elasticity of substitution decreases. This is because the more general form of Box-Cox transformations allows greater flexibility of time trends compared with single exponential trends. The explained variance is then redistributed among variables because of this increased flexibility, leading to changes in estimates of elasticity of substitution. The resulting production function is nearer to the Leontief function (LF)[8] than the Cobb-Douglas function (CDF), but is distinct from both limiting cases.

Table 1: Estimated results of the supply system

Description	Parameter \ Version	Private sector – exponential trend *Priv A*	Private sector – Box-Cox transformations *Priv B*	Public sector – exponential trend *Publ A*	Public sector – Box-Cox transformations *Publ B*
Constant term (PF)	A_0	0.971***	0.987***	0.973***	1***
Distribution parameter	a	0.6[c]	0.6[c]	0.3[c]	0.3[c]
Elasticity of substitution	σ	1.081***	0.304***	0.01[c]	0.127**
Rate of technical progress	ψ	0.026***		0.021***	
Parameter of trend for K (1)	γ_1		0.021***		0.018***
Parameter of trend for K (2)	λ_1		0.950***		-0.232
Parameter of trend for L (1)	γ_2		0.034***		0.026***
Parameter of trend for L (2)	λ_2		0.488***		0.071
Constant term (RD)	C_0	0.044*	-0.129**	-0.115***	-0.391***
AR term (relative demand)	ν	0.933***	0.784***	0.881***	0.539***
Base period		2006Q1	2006Q1	2004Q2	2004Q2
R^2 – prod. function		0.988	0.991	0.829	0.901
R^2 – relative demand		0.984	0.987	0.899	0.926
Log likelihood		-1586.6	-1570.3	-1423.1	-1382
Schwarz criterion		33.69	33.49	30.20	29.53

Note: Parameter marked with c are calibrated. The distribution parameter was calibrated with regard to shares of capital costs on value added. We find the use of common exponential trend in public sector inappropriate, but for the sake of comparison we estimate the system with calibrated elasticity of substitution as smallest value as possible, for which the system still holds.

Estimates of elasticity of substitution strongly depend on the specification of technical progress for the public sector as well. The value of the elasticity of substitution in Publ-A converges to zero, suggesting use of Leontief function instead. However, if we introduce separate efficiency trends, the elasticity of substitution rises above zero and the fit in both equations improves materially.

The evidence (supported by Schwarz criterion) suggests the superiority of using the specification B with efficiency trends (despite increased number of parameters). The resulting elasticity of substitution in public sector is lower than that for private sector, but still significantly different from zero. Results do not support the assumption of exponential technical progress in the public sector.

Efficiency gains

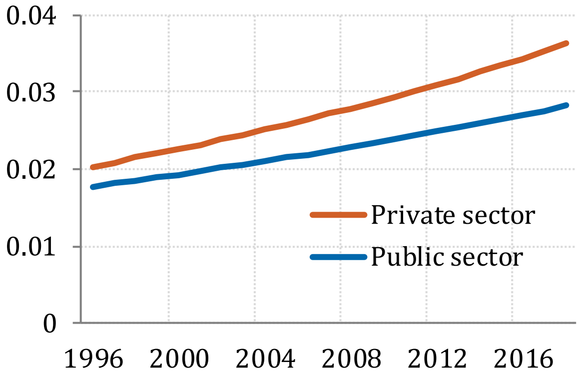
Technical progress causes a gradual rise of efficiency of production factors in time. It is modelled by functions of a linear time trend. We explored versions with single exponential

[7] Efficiency trend measures how efficiently is a relevant factor unit used, i.e. also as degree of progress.

[8] Leontief function refers to the complementary production function, as in Allen (1968).

trend that averages various effects and describes technical progress with a single parameter and more complex versions using separate trend functions for every production factor. We present the efficiency gains based on the trends from the production function.[9]

Figure 1: Exponential trends



Unsurprisingly, the technical progress is somewhat slower in the public sector. In both cases, the annual changes exhibit rising linear trend. The exponential function reflects the rise of a constant growth rate, but as results suggest, it can be too restrictive. The efficiency trends for supply systems with individual factor efficiency trends are presented in Figures 2a and 2b.

Figure 2a: Private sector efficiency trends

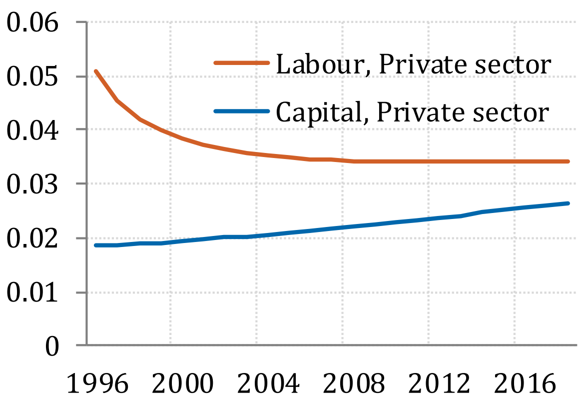
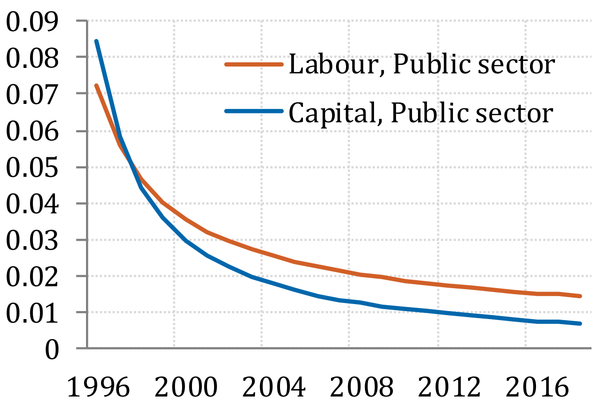


Figure 2b: Public sector efficiency trends



For both sectors, efficiency of labour grows faster than the efficiency of capital in most periods. The curves have distinctly different character from the previous case apart from capital in private sector. Although on slightly lower pace, efficiency of capital seems to accelerate over time (unlike in public sector). On the contrary, labour efficiency seems to grow roughly evenly over the past decade in private sector and by diminishing rate in public sector.

We see strictly decreasing efficiency gains over the early sample period, receding from very high efficiency growth levels in public sector.[10] Eventually, high gains in the private and public sectors in the early years (receding over time) along with subdued efficiency gains for

[9] The efficiency trends are equal to one in the base period (see Table 1).

[10] We have tested whether these high efficiency gains at the beginning of the sample could come as a result of data quality. We estimated the same system with sample starting in 1997 (not reported) and shifted the base year. The parameters changed marginally, but the shape of efficiency trend remained the same, thus we consider presented results to be robust.

capital in public sector could be a result of infrastructure being added to the capital stock as it is then mostly used for free or for minor charges.[11]

The analysis of eventual use of individual efficiency trends in both sectors provides only a suggestive evidence about size of the gains over time. However, it makes an evident case that technical progress is too complex phenomenon to be captured by CDF or by a simple exponential trend.

Summary

We present estimates of supply systems for Slovakia and construct annual efficiency gains for private and public sector based on estimated systems. Main findings are threefold:

1. We find that the differences of production process in public and private sector are so great that they need to be modelled separately.
2. The assumptions about technical progress impact the results greatly. Use of single exponential trend leads to elasticity of substitution equal to one in private sector (CDF) and zero in public sector (LF). The world of producing output however seems to be much more colourful than that. If individual efficiency trends are introduced, the elasticity of substitution for both sectors is significantly greater than zero, but below 0.5. Therefore, the use of a CES function is required. Separate efficiency trends derived from Box-Cox transformations are leading to better fit for both sectors. Application of single exponential trend is too restrictive.
3. The efficiency gains (annual changes of labour and capital efficiency derived from the model with Box-Cox transformation) increase or converge to a positive constant in the private sector, whereas they are strictly decreasing in the public sector.

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[11] Physical infrastructure as roads, bridges, but also technical infrastructure as ITC and/or transaction-based networks.

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